

Metric spaces

We will dedicate our remaining time to a particular type of topological spaces: metric spaces.

First of all, let's see what a topology is in general.

Let X be any set (ex: \mathbb{R} , \mathbb{R}^n , $[0,1]$, \mathbb{Q} , {banana, orange, apple}).

We make an arbitrary choice of sets \mathcal{U} in X , which we agree to call open if they satisfy certain, quite general, properties.

Once we decide which sets we will call open, we say that the family of these sets is a topology on X .

There is a whole theory (Topology) that helps us understand properties of X through our choice of open sets. Choosing different open sets can reveal different properties of X .

When there is a notion of ^(metric) distance in X (such as $| \cdot |$ in \mathbb{R}), there is a quite

(2)

natural choice of open sets in X (intervals)

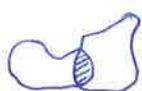
(a, b) in \mathbb{R} fall in that category). And this natural choice reveals a lot about X (even though other choices can reveal other things, e.g. weak topologies in functional Analysis).

It is this natural choice of open sets in a metric space X that this chapter will focus on.

Let us see precisely what a topology is on any set $X(\neq \emptyset)$:

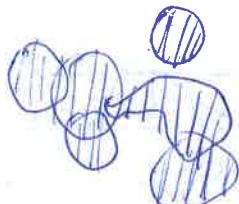
→ Def: Let $X \neq \emptyset$, a set. A family \mathcal{T} of subsets of X is called a topology on X if it satisfies the following:

i) $\emptyset, X \in \mathcal{T}$.



finite intersections of open sets are open ii) If $U_1, \dots, U_n \in \mathcal{T}$, then $U_1 \cap \dots \cap U_n \in \mathcal{T}$, then

arbitrary unions of open sets are open iii) If $U_i \in \mathcal{T} \forall i \in I$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$



an indexing set, it could be uncountable, like \mathbb{R} , or even larger

(3)

- Then, (X, τ) is called a **topological space**, and the elements of τ are called the **open sets** in X with respect to the topology τ .

- We call **closed** every $f \subseteq X$ s.t. $X \setminus f$ is open.
(So, \emptyset and X are open and closed!).
- We call a **basis** for the topology τ

any subfamily τ' of τ s.t. :

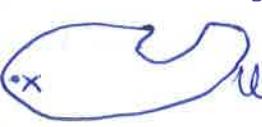
i.e., if any open U is a union of elements of τ' .
if $U \in \tau$, then $U = \bigcup_{i \in I} U_i$, where the U_i 's are in τ' .



(i.e., a **basis** for τ is a subcollection of open sets, such that their unions generate all open sets)



⚠ A basis of τ is all we need to know to get τ . So, many times, when we want to show something for open sets, it suffices to show it for all elements of a basis.

- A **neighbourhood** of $x \in X$ is any $U \in \tau$ s.t. $x \in U$ (i.e., any open set that contains x). 
- Let $x \in X$. A **basis of neighbourhoods of x** is any

(4)

collection N_x of neighbourhoods of x , s.t. :

if V is a neighbourhood of x , then there exists $U \in N_x$ s.t. $x \in U \subseteq V$.



⚠

Again, the idea is that, in general, if we want to show something for any neighbourhood of x , it will suffice to show it only for the neighbourhoods in a basis of neighbourhoods of x .

⚠

We will eventually see that open intervals in \mathbb{R} form a basis for the "usual" topology in \mathbb{R} . And intervals of the form $(x-\epsilon, x+\epsilon)$, $\forall \epsilon > 0$, form a basis of neighbourhoods of x .

→ Two very basic examples: Let $X \neq \emptyset$ be a set.

- $T = \{\emptyset, X\}$ is a topology on X . It is called the trivial topology. X is the only neighbourhood of any $x \in X$.

- $T = \{U : U \subseteq X\}$ (the set of all subsets of X)

is a topology on X . It is called the discrete topology. (because $\{x\}$ is open, $\forall x \in X$).

The singletons, together with \emptyset ,

form a basis for the topology. For each $x \in X$,

(5)

$\{x\}$ forms a basis of the neighbourhoods of x .

All non-empty subsets of X that contain x are neighbourhoods of x .

All subsets of X are open (as they are in T).
And all subsets of X are closed.

→ More examples: Let $X = \{a, b, c\}$. The families

$$T = \{\emptyset, \{a, b, c\}, \{a, b\}\}$$

$$\text{and } T' = \{\emptyset, \{a, b, c\}, \{c\}\}$$

are both topologies on X .

$\{a, b\}$ is open w.r.t. T , and closed w.r.t. T' .

$\{b\}$ is not open not closed, w.r.t. any of the two topologies.

Metric spaces are very particular cases of topological spaces.

To start with, metric spaces are spaces with a notion of distance in them.

for example, take \mathbb{R} . There is a notion of distance in \mathbb{R} : for each $x, y \in \mathbb{R}$, $|x-y|$ is the distance between x and y .

(6)

Note that this distance in \mathbb{R} satisfies 3 basic properties:

- (i) $|x-y| \geq 0$ $\forall x, y \in \mathbb{R}$, and $|x-y|=0 \iff x=y$.
- (ii) $|x-y|=|y-x|$, $\forall x, y \in \mathbb{R}$. (symmetry).
- (iii) $|x-y| \leq |x-z| + |z-y|$, $\forall x, y, z \in \mathbb{R}$ (triangle inequality).

Any function $d: X \times X \rightarrow \mathbb{R}$ that satisfies the above properties can be thought of as a distance on X , and is called a metric on X :

\rightarrow Def: Let $X \neq \emptyset$, a set. A function $d: X \times X \rightarrow \mathbb{R}$ is called a **metric on X** if it satisfies the following properties:

- (i) $d(x, y) \geq 0$ $\forall x, y \in X$, and $d(x, y)=0 \iff x=y$.
- (ii) $d(x, y) = d(y, x)$, $\forall x, y \in X$. (symmetry)
- (iii) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$ (triangle inequality).



If d is a metric on X , we say that X is a metric space w.r.t. d , and we denote it by (X, d) (to indicate which metric we are considering).

(7)

→ Examples:

① The function $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$,

will induce
usual topology
on \mathbb{R} (w.r.t.
which (a, b)
will be open).

with $d(x, y) = |x - y|, \forall x, y \in \mathbb{R}$,

is a metric on \mathbb{R} . It is known as the usual metric on \mathbb{R} .

② Let $X \neq \emptyset$, a set. Then, the function $d: X \times X \rightarrow \mathbb{R}$,

such as \mathbb{R}
with $d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}, \forall x, y \in X$,

is a metric on X .

(Note that each x is at distance 1 from all other elements of X .)

This is known as the discrete metric on X
(because it will induce the discrete topology).

③ Let $X = \mathbb{R}^n$. The function $d: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$,

will induce
the usual
topology on \mathbb{R}^n
with $d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}, \forall x, y \in \mathbb{R}^n$,

is a metric on \mathbb{R}^n . It is known as the usual

metric on \mathbb{R}^n . We denote this metric by d_2 .

(8)

4 Let $p \geq 1$, and $X = \mathbb{R}^n$. The function $d: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$,
 with
$$d(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{1/p}, \quad \forall x, y \in \mathbb{R}^n,$$

 is a metric on \mathbb{R}^n . We denote this metric by d_p .

"Proof" that d_p satisfies the triangle inequality, $\forall p \geq 1$:

It is based on Minkowski's inequality:

If $p \geq 1$, then, $\forall a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$, we have

$$\left(\sum_{i=1}^n |a_i + b_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^n |a_i|^p \right)^{1/p} + \left(\sum_{i=1}^n |b_i|^p \right)^{1/p}.$$

We will not see a proof of this.

i.e.: ("length" of vector $[a + b] \in \mathbb{R}^n$) \leq ("length" of a) + ("length" of b). □

Since, $\forall i = 1, \dots, n$, $x_i - y_i = (x_i - z_i) + (z_i - y_i)$,

we have
$$\left(\sum_{i=1}^n |x_i - y_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^n |x_i - z_i|^p \right)^{1/p} + \left(\sum_{i=1}^n |z_i - y_i|^p \right)^{1/p}$$

$$\stackrel{(x_i - z_i) + (z_i - y_i)}{\leq} \stackrel{||}{a_i} + \stackrel{||}{b_i}$$

⚠ Notice that the usual metric on \mathbb{R} equals d_p , $\forall p \geq 1$.
 The usual metric on \mathbb{R}^2 equals d_2 (on \mathbb{R}^2)

(9)

Let's generalise d_p to any Cartesian product (of finitely many metric spaces):

(5) Let (X_i, d_i) be metric spaces, $\forall i = 1, \dots, n$.

Let $X = X_1 \times \dots \times X_n$ ($= \{(x_1, \dots, x_n) : x_1 \in X_1, x_2 \in X_2, \dots, x_n \in X_n\}$).

Let $p \geq 1$.

Then, the function $d_p : X \times X \rightarrow \mathbb{R}$,

$$\text{with } d_p(x, y) = \left[\sum_{i=1}^n (d_i(x_i - y_i))^p \right]^{1/p}, \quad \forall x, y \in X,$$

is a metric on X .

Proof:



from now on, whenever we refer to $X_1 \times \dots \times X_n$ as a metric space, we will mean with metric d_p , unless stated otherwise!

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in X .

$\forall i = 1, \dots, n, \quad x_i - y_i = (x_i - z_i) + (z_i - y_i)$

triangle inequality
for d_i

$$d_i(x_i - y_i) \leq d_i(x_i - z_i) + d_i(z_i - y_i).$$

$$\text{So, } d_p(x, y) = \left[\sum_{i=1}^n (d_i(x_i - y_i))^p \right]^{1/p} \stackrel{*}{\leq} \left[\sum_{i=1}^n \left(\underbrace{d_i(x_i - z_i)}_{d_i} + \underbrace{d_i(z_i - y_i)}_{d_i} \right)^p \right]^{1/p}$$

Minkowski's
inequality

$$\left[\sum_{i=1}^n (d_i(x_i - z_i))^p \right]^{1/p} + \left[\sum_{i=1}^n (d_i(z_i - y_i))^p \right]^{1/p} = d_p(x, z) + d_p(z, y).$$

Lecture 33:

①

Normed spaces:

Certain metrics on vector spaces

are induced by norms:

(for instance, d_p on \mathbb{R}^n , $n \geq 1$; and, in particular, the usual metric on \mathbb{R}^n)

so that $x+y, \alpha x \in X$, whenever $x, y \in X, \alpha \in \mathbb{R}$!

(or else...)

Otherwise,
(ii) and (iii)
would make no
sense.

Def: Let X be a vector space (with scalars in \mathbb{R}).

A function $\| \cdot \| : X \rightarrow \mathbb{R}$,

with $x \mapsto \|x\|, \forall x \in X$,

is called a norm on X if

(i) $\|x\| \geq 0, \forall x \in X$, and $\|x\| = 0 \iff x = 0$.

(ii) $\|\alpha x\| = |\alpha| \cdot \|x\|, \forall x \in X, \forall \alpha \in \mathbb{R}$.

(iii) $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in X$ (triangle inequality).

(2)

→ Examples:

① for $X = \mathbb{R}$ (a vector space with scalars in \mathbb{R});

the absolute value $| \cdot | : \mathbb{R} \rightarrow \mathbb{R}$ is a norm.

② for $X = \mathbb{R}^n, n \in \mathbb{N}$ (a vector space with scalars in \mathbb{R});

the function $\| \cdot \|_2 : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\text{with } \|x\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \quad (= \sqrt{x_1^2 + \dots + x_n^2})$$

$x = (x_1, \dots, x_n) \quad \forall x \in \mathbb{R}^n$

is a norm on \mathbb{R}^n (for $n=1$, it equals $| \cdot |$).

We call it the usual norm in \mathbb{R}^n .

Notice that $d_2(x, y) = \|x - y\|_2, \forall x, y \in \mathbb{R}^n$.

③ More generally, for $p \geq 1$:

The function $\| \cdot \|_p : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\text{with } \|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}, \quad \forall x \in \mathbb{R}^n,$$

is a norm on \mathbb{R}^n . (Δ Not a norm for $p < 1$, as triangle inequality is not satisfied!)

Notice that $d_p(x, y) = \|x - y\|_p, \forall x, y \in \mathbb{R}^n$.

(3)

Notice that the triangle inequality for $\|\cdot\|_p$ is
exactly

Minkowski's inequality :

$$\left(\sum_{i=1}^n |x_i + y_i|^p \right)^{1/p} \leq \underbrace{\left(\sum_{i=1}^n |x_i|^p \right)^{1/p}}_{\|x\|_p} + \underbrace{\left(\sum_{i=1}^n |y_i|^p \right)^{1/p}}_{\|y\|_p}$$

$\|x+y\|_p$
 $(x_1, \dots, x_n) \quad \parallel \quad (y_1, \dots, y_n)$

$\forall x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}^n$

is the same as saying that

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p, \quad \forall x, y \in \mathbb{R}^n.$$

This is perhaps the nicest notation for
Minkowski's inequality.

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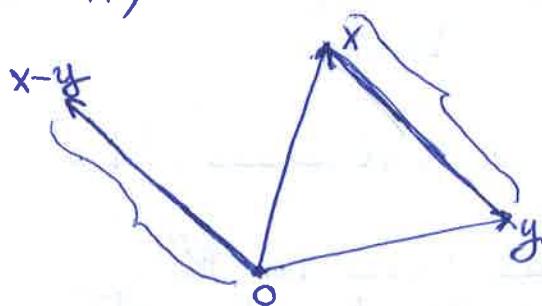
The idea now is that each norm $\|\cdot\|$ on a vector space X can induce a metric on X , by

setting the distance between any two vectors

$x, y \in X$ to be $\|x-y\|$.

It has to be proved that this is indeed a metric on X . Before we do this, notice the following:

If we indeed see $\|x-y\|$ as the distance of x from y in X ,



then $x-y = (x-y) - 0$ implies
 $\|x-y\| = \|(x-y) - 0\|$,

so we should have that the distance of x from y is the same as the distance of $x-y$ from 0 .

Similarly, $\|x-y\| = \|(x+a)-(y+a)\|$, the distance of $x+a$ from $y+a$. So, this $\|x-y\|$ distance is preserved under translations. This is not necessarily true for distances that are not induced by norms.

(5)

→ Prop: Let $(X, \|\cdot\|)$ be a normed (vector) space.

Then, the norm $\|\cdot\|$ induces a metric $d_{\|\cdot\|}$ on X :

the function $d_{\|\cdot\|}: X \times X \rightarrow \mathbb{R}$,

$$\text{with } d_{\|\cdot\|}(x, y) = \|x - y\|, \quad \forall x, y \in X.$$

(Δ And therefore $(X, d_{\|\cdot\|})$ is a metric space).

Proof: We need to show that the three properties of a metric are satisfied by $d_{\|\cdot\|}$.

Indeed, $d_{\|\cdot\|}: X \times X \rightarrow \mathbb{R}$ satisfies:

$$(i) \boxed{d_{\|\cdot\|}(x, y) \geq 0 \quad \forall x, y \in X, \text{ and } d_{\|\cdot\|}(x, y) = 0 \iff x = y :}$$

$d_{\|\cdot\|}(x, y) = \|x - y\|$, which is $\geq 0 \quad \forall x, y \in X$
(by property (i) of norm),

and $= 0 \iff x - y = 0 \iff x = y$
(again, by property (i) of norm).

$$(ii) \boxed{d_{\|\cdot\|}(x, y) = d_{\|\cdot\|}(y, x), \quad \forall x, y \in X :}$$

$$d_{\|\cdot\|}(x, y) = \|x - y\| = \|y - x\| = d_{\|\cdot\|}(y, x), \quad \forall x, y \in X.$$

property
(ii) of norm,
for $\alpha = -1$:

$$(-1)(x - y) = y - x, \text{ so } \|y - x\| = \|(-1)(x - y)\| = |(-1)| \cdot \|x - y\| = \|x - y\|$$

(6)

$$(iii) \boxed{d_{\|\cdot\|} (x, y) \leq d_{\|\cdot\|} (x, z) + d_{\|\cdot\|} (z, y), \forall x, y, z \in X:}$$

$$d_{\|\cdot\|} (x, y) = \|x - y\| \leq \|x - z\| + \|z - y\| = d_{\|\cdot\|} (x, z) + d_{\|\cdot\|} (z, y),$$

triangle
 inequality for
 norm, as
 $x - y = (x - z) + (z - y)$

→ Application: • The usual metric on \mathbb{R} is induced by
 $d_\infty = d_p, (\forall p \geq 1, \dots)$

the norm $|\cdot|$. (So, one doesn't need to specially prove that the usual metric on \mathbb{R} is a metric; it follows by the fact that the absolute value is a norm).

- The usual metric d_∞ on \mathbb{R}^n is induced by $\|\cdot\|_\infty$ on \mathbb{R}^n (as $d_\infty(x, y) = \|x - y\|_\infty, \forall x, y \in \mathbb{R}^n$).

So, it follows directly that d_∞ is a metric.

- The metric d_p on \mathbb{R}^n is induced by the norm $\|\cdot\|_p$ on \mathbb{R}^n (as $d_p(x, y) = \|x - y\|_p, \forall x, y \in \mathbb{R}^n$).

So, it follows directly that d_p is a metric.

Metrics induced by norms are very restrictive: (7)



Let $X \neq \emptyset$, a set. NOT every metric d on X is induced by a norm on X :

Let $(X, \|\cdot\|)$ be a normed space. Then:

• X is a vector space (otherwise we can't talk about a norm on X).

• $d_{\|\cdot\|}$ is translation invariant

$$\text{i.e., } d_{\|\cdot\|}(x+a, y+a) = d_{\|\cdot\|}(x, y), \forall a \in X.$$

This has already been explained, easy.

If $X \neq \{0\}$, $d_{\|\cdot\|}$ takes all non-negative values (i.e., full range $\mathbb{R}_{\geq 0}$)

(i.e., $\forall c \geq 0$ in \mathbb{R} , we have that c

is materialised as the distance $d_{\|\cdot\|}$ between two vectors $x, y \in X$).

Proof: Let $v \neq 0$ be a vector in X .

Then, $\|v\| \neq 0 \rightarrow \|av\| = |a| \cdot \|v\|$ takes all

values in $\mathbb{R}_{>0}$ as a runs in $[0, \infty)$.

as we take all multiples av of v , their distance runs from 0 to ∞ .

i.e., any non-negative number is the norm of some vector, i.e. its distance $d_{\|\cdot\|}$ from 0.

8

→ Example: The discrete metric, $d: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\text{with } d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

is NOT induced by a norm on \mathbb{R} .

Indeed, d only takes two values (rather than having all the non-negative numbers in its range).

(9)

→ Topology induced by a metric:

let (X, d) be a metric space. The metric d induces a "natural" topology T_d on X , as follows:

- We put \emptyset in T_d .

- We put in T_d all sets

Δ By definition, each $B_d(x_0, r)$ is a subset of X !

Δ Changes if d changes!

$$B_d(x_0, r) := \{x \in X : d(x, x_0) < r\},$$

over all $x_0 \in X$ and $r > 0$.

Since $B_d(x_0, r)$ consists of all elements of X

at distance at most r from x_0 , we say

that $B_d(x_0, r)$ is a ball, with centre x_0 and radius r .

In fact, we call $B_d(x_0, r)$ the open ball with

centre x_0 and radius r , as it is in T_d , and thus

is an open set w.r.t. T_d , the topology

induced by the metric $\left(\text{later, we prove that } T_d \text{ is a topology, justifying our convention to call these balls open.} \right)$

However, there may be metrics on X w.r.t.

which $B_d(x_0, r)$ is not a ball, or even an open set.

So, even though $B_d(x_0, r)$ is called an "open ball",

we should remember that it is a ball, and open, w.r.t.

d ; not w.r.t. any metric (or topology) on X .

we want

- Since T_d to be a topology containing all open balls w.r.t. d , it has to contain all their unions, too.

In fact, we define

$$T_d := \{ \phi, \text{ all unions of open balls in } X \}.$$

(i.e., $T_d = \{ \mathcal{U} \subseteq X : \mathcal{U} = \phi \text{ or } \mathcal{U} = \bigcup_{i \in I} B(x_i, r_i) \}$,

where I is any indexing set
and $x_i \in X, r_i > 0, \forall i \in I \}$.)

[Really, for ϕ we can use $I = \emptyset \dots$]

It is not very clear yet that T_d is a topology.

To prove it, we first find a nice characterisation of the sets in T_d (which will be the open sets w.r.t. T_d).

once we prove that T_d is a topology...

Characterisation of sets in T_d (eventually, of open sets w.r.t. d)

Let $\mathcal{U} \subseteq X$, $\mathcal{U} \neq \emptyset$.
 (X, d)

Then,

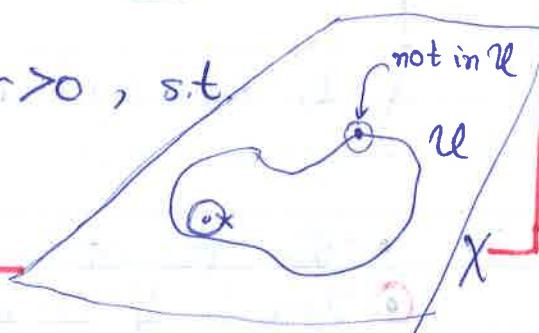
$$U \in T_d$$

↔

For $x \in U$, there exists $r > 0$, s.t.

$$B(x, r) \subseteq U$$

Really, this says that
 U doesn't contain its
boundary...
 in X .

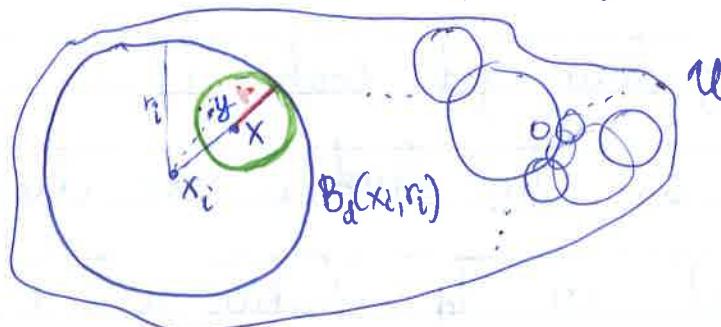


Proof: (\Rightarrow) ~~$U \in T_d$~~ . Therefore, by definition of T_d ,
 U is a union of open balls. I.e.:

$$U = \bigcup_{i \in I} B_d(x_i, r_i), \quad \begin{array}{l} \text{for some indexing set } I, \\ \text{for some } x_i \in X, r_i > 0 \\ i \in I. \end{array}$$

let $x \in U$. Then, $x \in B_d(x_i, r_i)$, for some $i \in I$.

And of course $B_d(x_i, r_i) \subseteq U$ (as it is one of the balls whose union is U).



(12)

We will show that there exists open ball

$$B_d(x, r) \subseteq B_d(x_i, r_i) \quad ; \text{ thus,}$$

$B_d(x, r)$ will be an open ball centered at x , fully contained in U .

Indeed, let $r := r_i - d(x, x_i)$

We have : $B_d(x, r) \subseteq B_d(x_i, r_i)$.

Indeed, let $y \in B_d(x, r)$. We will show that $y \in B_d(x_i, r_i)$, i.e. that $d(y, x_i) < r_i$:

$$d(y, x_i) \leq \underbrace{d(y, x) + d(x, x_i)}_{\substack{\text{triangle} \\ \text{inequality}}} < r_i \quad \text{as } y \in B_d(x, r)$$

$$\leq r + d(x, x_i) = r_i - d(x, x_i) + d(x, x_i) = r_i$$

by definition
of r

thus $y \in B_d(x_i, r_i)$.

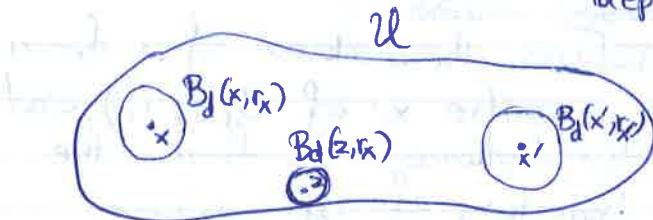
(13)

(\Leftarrow) We want to show that our non-empty U can be written as a union of open balls.

Indeed, we know that, for each $x \in U$, there

exists an open ball $B_d(x, r_x)$, s.t. $B_d(x, r_x) \subseteq U$.

$B_d(x, r_x)$
centered at x ,
of some radius
 $r_x > 0$ (that
depends on x)



$$\text{Thus, } U = \bigcup_{x \in U} B_d(x, r_x)$$

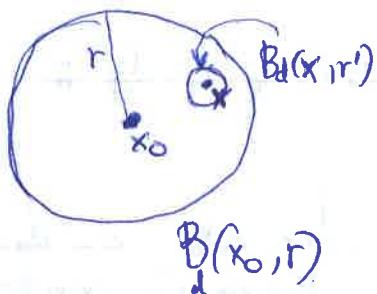
Since $B_d(x, r_x) \subseteq U \quad \forall x \in U$, then $\bigcup_{x \in U} B_d(x, r_x) \subseteq U$

And $x \in B_d(x, r_x) \quad \forall x \in U$, thus

$$U \subseteq \bigcup_{x \in U} B_d(x, r_x)$$

So, $U \in T_d$.

⚠ Note that this Proposition also applies for the open balls, as they belong to T_d . In particular, for any $x_0 \in X, r > 0$, we have that: $\forall x \in B_d(x_0, r)$, there exists $r' > 0$ s.t. $B_d(x, r') \subseteq B_d(x_0, r)$:



And, of course, this

works for $r' = r - d(x, x_0)$ (see proof),

just like our intuition directs us to expect (this r' is the distance of x from the boundary of the big ball $B_d(x_0, r)$); even though we haven't defined what a "boundary" is yet, our intuition tells us that this r' should work, and it does.)

→ Prop: T_d is a topology on X .

Proof: We want to show that T_d satisfies the three properties of a topology:

(i) $\emptyset \in T_d$ (by definition of T_d),

and $X \in T_d$, because $X = \bigcup_{x \in X} B_d(x, 1)$;

so X is a union of open balls w.r.t. d .

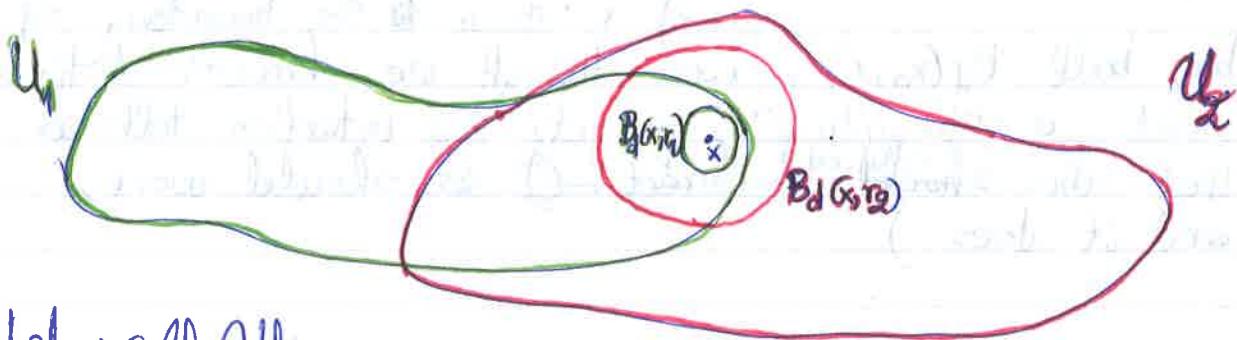
also, $X = \bigcup_{r>0} B_d(x_0, r)$,
for any fixed $x_0 \in X$!

(ii) finite intersections of sets in T_d are also in T_d . Indeed:

(15)

Let $U_1, U_2 \in T_d$. Then: $U_1 \cap U_2 \in T_d$:

(by our characterisation of sets in T_d) It suffices to show that, for all $x \in U_1 \cap U_2$, there exists an open ball $B_d(x, r)$ centered at x , such that $B_d(x, r) \subseteq U_1 \cap U_2$.



Let $x \in U_1 \cap U_2$.

We know that:

- $x \in U_1 \in T_d \Rightarrow$ there exists an open ball $B_d(x, r_1)$, centered at x , s.t. $B_d(x, r_1) \subseteq U_1$.
- $x \in U_2 \in T_d \Rightarrow$ there exists an open ball $B_d(x, r_2)$, centered at x , s.t. $B_d(x, r_2) \subseteq U_2$.

So, for $r = \min\{r_1, r_2\}$, we have that

$$B_d(x, r) \subseteq U_1 \cap U_2.$$

Since $x \in U_1 \cap U_2$ was arbitrary, $U_1 \cap U_2 \in T_d$.

By induction, it follows that finite intersections of sets in T_d are in T_d .

(16)

(iii) Arbitrary unions of sets in T_d are in T_d .

Indeed: Let $U_i \in T_d$, $i \in I$

I is an arbitrary indexing set.

Then, each U_i is a union of open balls w.r.t. d ,

so $\bigcup_{i \in I} U_i$ is a union of open balls w.r.t. d ,

thus $\bigcup_{i \in I} U_i$ is in T_d . ■

(16)

(iii) Arbitrary unions of sets in T_d are in T_d .

Indeed: Let $U_i \in T_d$, $i \in I$

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Then, each U_i is a union of open balls w.r.t. d ,

so $\bigcup_{i \in I} U_i$ is a union of open balls w.r.t. d ,

thus $\bigcup_{i \in I} U_i$ is in T_d .

(1)

Lecture 34:

→ Def: Now that we have shown that T_d is a topology,

we will be calling all elements of T_d

(i.e., all open sets w.r.t. T_d) open sets w.r.t. d ,

or open subsets of (X, d) , or open subsets of X ,

if it is clear that we are using d as the metric on X , and not another metric

(2)

 Notice that, by proving that T_d is a topology,

we make sense of our initial decision to call the balls $B_d(x, r)$ "open". Up to now, this was just a convention.

Now, we know that all balls $B_d(x, r)$ are open sets with respect to T_d , and thus, by definition, are open subsets of X , w.r.t. the metric d .



Let us list the things we have proved:

Each metric d on $X (\neq \emptyset)$ induces a topology T_d on X .

With respect to this topology:

- Any ball $B_d(x, r)$ is an open subset of X .
- Any $\bigcup_{U \in \mathcal{U}} U$ subset of X is open



$\forall x \in X, \forall U \in \mathcal{U}, \text{ there exists an open ball } B_d(x, r) \subseteq U$.

- The open balls w.r.t. d form a basis for the topology T_d (as any $\neq \emptyset$ open set is the union of open balls).
- $\forall x \in X, \text{ the open balls centered at } x \text{ form a basis of neighbourhoods of } x$ (as $\forall U$ a neighbourhood

(3)

of x in T_d , i.e., for any open $\mathcal{U} \ni x$, we know that there exists an open ball $B(x, r)$ centered at x s.t. $B(x, r) \subseteq \mathcal{U}$. $\boxed{\text{a neighbourhood of } x, \text{ as it is in } T_d \text{ and contains } x}$

Remember, the closed subsets of (X, d) are the subsets F of X s.t. $X \setminus F$ is open (in (X, d)).

Prop: Let (X, d) be a metric space. Then, for any $x_0 \in X$ and $r \geq 0$, $\boxed{\text{We are allowing } r=0, B_d(x_0, 0) = \{x_0\}}$

the set $\tilde{B}_d(x_0, r) := \{x \in X : d(x, x_0) \leq r\}$ is closed in (X, d) .

! So, singletons are closed
w.r.t. any metric!

We will thus call it the closed ball with centre x_0 and radius r .

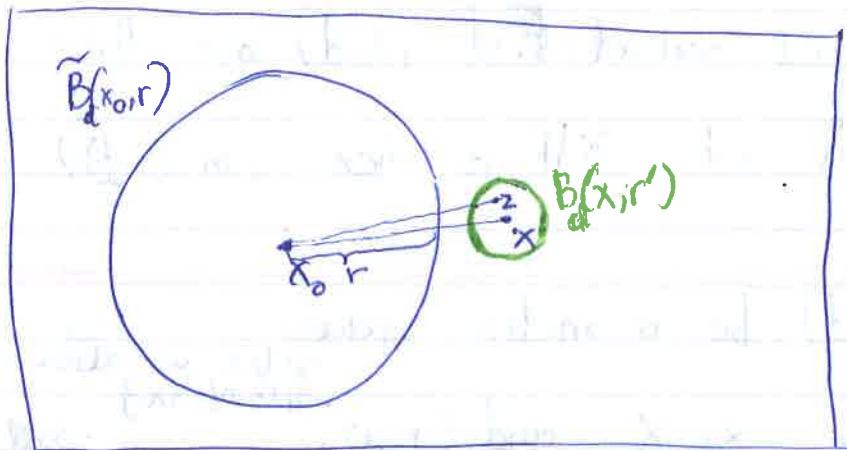
Proof: Let $x_0 \in X$, $r > 0$. To show that $\tilde{B}_d(x_0, r)$ is closed in (X, d) , we must show that $X \setminus \tilde{B}_d(x_0, r)$ is open in (X, d) .

Notice that $X \setminus \tilde{B}_d(x_0, r) = \{x \in X : x \notin \tilde{B}_d(x_0, r)\} = \{x \in X : d(x, x_0) > r\}$.

(4)

Let $x \in X \setminus \tilde{B}_d(x_0, r)$

We will show that there exists some open ball in (X, d) , centered at x , fully contained in $X \setminus \tilde{B}_d(x_0, r)$:



We feel that we should be able to make the radius r' as large as $d(x_0, x) - r$ before we touch the original ball $B_d(x_0, r)$. Indeed, this is true.

Indeed, let $r' := d(x_0, x) - r$ (≥ 0 as $x \notin \tilde{B}_d(x_0, r)$).

Then, $B_d(x, r') \subseteq X \setminus \tilde{B}_d(x_0, r)$:

Let $z \in B_d(x, r')$. Then,

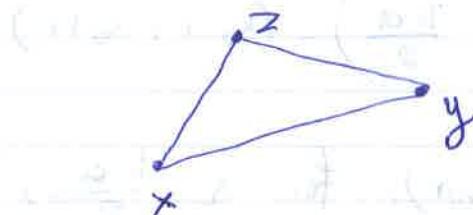
$$d(z, x_0) \geq d(x_0, x) - d(z, x) \quad (\text{by triangle inequality})$$

$$\geq d(x_0, x) - r' = \cancel{d(x_0, x)} - \cancel{d(x_0, x)} + r = r, \\ d(x, z) < r'$$

so $z \notin \tilde{B}_d(x_0, r)$, thus $z \in X \setminus \tilde{B}_d(x_0, r)$.

(5)

 In the above proof, we see a useful way to think of the triangle inequality:



It doesn't just say that each side of the triangle has length at most the sum of the lengths of the other two sides.

It also says that each side has length at least the difference of the lengths of the other two sides:

$$d(x, y) \leq d(x, z) + d(z, y) \implies d(z, y) \geq d(x, y) - d(x, z).$$

→ Application on \mathbb{R} , equipped with the usual metric

$$d : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad d(x, y) = |x - y|, \quad \forall x, y \in \mathbb{R}:$$

- For any $x_0 \in \mathbb{R}$, $r > 0$,

$$\begin{aligned} B_d(x_0, r) &= \{x \in \mathbb{R} : d(x, x_0) < r\} = \{x \in \mathbb{R} : |x - x_0| < r\} = \\ &= (x_0 - r, x_0 + r). \end{aligned}$$



(6)

- Also, notice that any interval (a, b) , with $a, b \in \mathbb{R}$, is an open ball with respect to the usual metric:

$$(a, b) = \left(\frac{b-a}{2} - \frac{b-a}{2}, \frac{b+a}{2} + \frac{b-a}{2} \right) = (x_0 - r, x_0 + r) =$$
$$= B_d(x_0, r), \text{ for } x_0 = \frac{b+a}{2}, r = \frac{b-a}{2}.$$

So, all the open balls in \mathbb{R} w.r.t. the usual metric are all the open intervals (a, b) in \mathbb{R} , with $a, b \in \mathbb{R}$. (And they are all open sets w.r.t. the metric.)

- So, any $\emptyset \neq \mathcal{U} \subseteq \mathbb{R}$ is open w.r.t. the usual metric

$\Rightarrow \mathcal{U}$ is a union of open intervals (with finite endpoints).

This follows by the definition of the topology induced by the metric.

- Moreover, any $\emptyset \neq \mathcal{U} \subseteq \mathbb{R}$ is open w.r.t. the usual metric

$\Leftrightarrow \forall x \in \mathcal{U}$, there exists a neighbourhood $(x-r, x+r)$ of x that is contained in \mathcal{U} .

(7)

This follows by the Proposition we proved, that

$$U \text{ open} \iff \forall x \in U \exists B_d(x, r) \subseteq U.$$

for any $x_0 \in \mathbb{R}$, the set $\{(x_0 - r, x_0 + r) : r > 0\}$

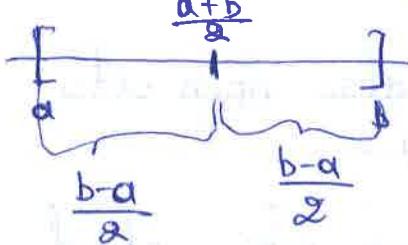
is a basis of neighbourhoods of x_0 .

• for any $x_0 \in \mathbb{R}$, $r \geq 0$,

$$\begin{aligned}\tilde{B}_d(x_0, r) &= \{x \in \mathbb{R} : d(x, x_0) \leq r\} = \{x \in \mathbb{R} : |x - x_0| \leq r\} = \\ &= [x_0 - r, x_0 + r]. \quad (\text{for } r=0, \text{ this is the singleton } \{x_0\}).\end{aligned}$$

Also, notice that any interval $[a, b]$, with $a \leq b$ in \mathbb{R} , is a closed ball w.r.t. the usual metric:

$$\begin{aligned}[a, b] &= \left[\frac{a+b}{2} - \frac{b-a}{2}, \frac{a+b}{2} + \frac{b-a}{2} \right] = [x_0 - r, x_0 + r] = \\ &= \tilde{B}_d(x_0, r), \text{ for } x_0 = \frac{a+b}{2}, r = \frac{b-a}{2} \geq 0.\end{aligned}$$



including singletons

So, all intervals $[a, b]$, with $a \leq b$ in \mathbb{R} are closed sets wrt. the usual metric

(8)

(because all closed balls w.r.t. a metric are closed sets w.r.t. the topology induced by the metric.)

This is why these intervals are actually called closed intervals.

→ Application to (X, d) , where d is the discrete metric $d: X \times X \rightarrow \mathbb{R}$

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases} :$$

- for any $x_0 \in X$, there are no points in X at distance < 1 from x_0 , apart from x_0 .

Thus, $\forall r \in (0, 1)$, $B_d(x_0, r) = \{x_0\}$. \rightarrow each singleton is an open ball.

Thus, any singleton is an open set w.r.t. d .

So, all unions of singletons are also open sets w.r.t. d (property (iii) of topology),

i.e. all subsets of X are open sets w.r.t. d .

(9)

This means that

$$T_d = \{ \text{all subsets of } X \}, \text{ the discrete topology.}$$

i.e.: the topology induced by the discrete metric is the discrete topology.

W.r.t. the discrete metric d , all subsets of X are open and closed.

- We saw that, for all $x_0 \in X$,

$$\text{if } r \in (0,1), \text{ then } B_d(x_0, r) = \{x_0\}.$$

On the other hand, all elements of X , apart from x_0 , are at distance 1 from x_0 .

$$\text{So, for all } r \geq 1, B_d(x_0, r) = X.$$

Thus: all balls of radius < 1 are singletons,

and all balls of radius ≥ 1 are the same: the whole set X .

(so, e.g.: in \mathbb{R} with the discrete topology, $(x_0 - 1, x_0 + 1)$ is not an open ball).

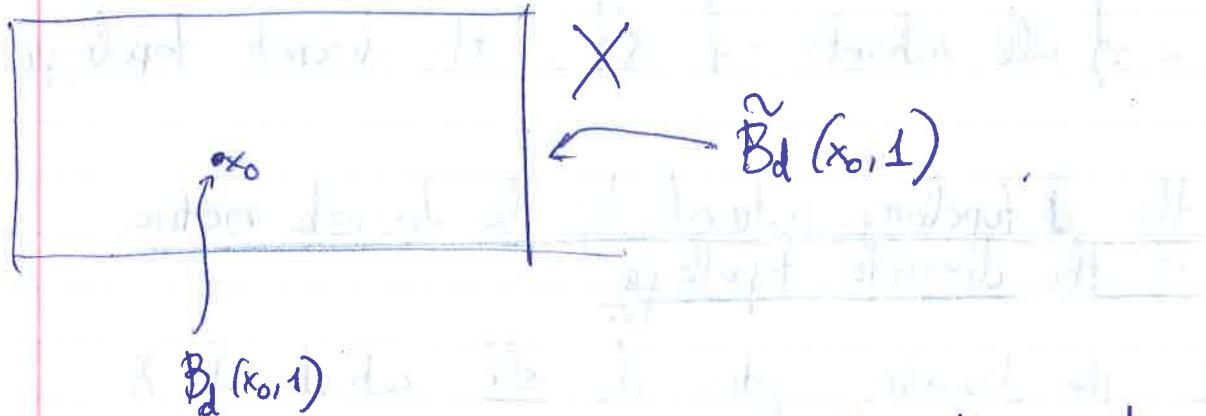
i.e., there is no ball that contains any number of points between 1 and the cardinality of X .

- Notice that $B_d(x_0, 1) = \{x_0\}$, while $\overline{B_d(x_0, 1)} = X$.

So, a closed ball can differ from the open

(10)

ball with the same centre and radius by a huge amount.



thus, this is true for metric spaces (X, d) , too.

Prop: Let (X, T) be a topological space. Then:

(i) finite unions of closed sets are closed (w.r.t. T)

(ii) arbitrary intersections of closed sets are closed (w.r.t. T).

Proof:

(i) Let $n \in \mathbb{N}$, and f_1, \dots, f_n closed subsets of X (w.r.t. T). Then, $X \setminus f_1, \dots, X \setminus f_n$ are open subsets of X .

And: $X \setminus (f_1 \cup \dots \cup f_n) = (X \setminus f_1) \cap \dots \cap (X \setminus f_n)$, open,

as a finite intersection of open sets.
So, $f_1 \cup \dots \cup f_n$ closed.

(11)

(ii) Let f_i be closed in X w.r.t. T , the $\{$

an arbitrary
indexing set.

Then, $X \setminus \left(\bigcap_{i \in I} f_i \right) = \bigcup_{i \in I} \underbrace{(X \setminus f_i)}_{\text{open}}$ open, as an arbitrary union of open sets.

So, $\bigcap_{i \in I} f_i$ is closed.

→ Corollary: In any metric space, all finite sets are closed (w.r.t. the metric).

Proof: Let (X, d) be a metric space.

Each singleton is closed (as a closed ball with radius 0).

So, by the Proposition above, finite unions of singletons are also closed.



Arbitrary intersections of open sets are not necessarily open! (while finite intersections and arbitrary unions are).

For instance, in the metric space \mathbb{R} , with the usual metric, $(-\frac{1}{n}, \frac{1}{n})$ is open for all $n \in \mathbb{N}$ (we showed that open

(12)

intervals are open w.r.t. the usual metric).

And $\bigcap_{i=1}^{+\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$, which is

not open in \mathbb{R} (as it is not a union of open intervals).

→ Closed sets in more detail:

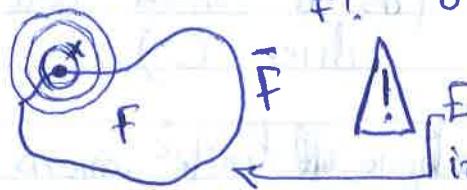
→ Def: Let (X, d) be a metric space. Let $F \subseteq X$.

The closure of F is defined as the set

$$\bar{F} := \{x \in X : F \cap B_d(x, r) \neq \emptyset, \forall r > 0\}$$

↓
not necessarily
 F !

no matter how close
we look at x , we find
elements of F !



⚠ Every open ball centered at x
intersects F . So, $x \in \bar{F}$.

The elements of \bar{F} are literally the points of X that arbitrarily close to F .

→ Observation:

$$\bar{\emptyset} = \emptyset.$$

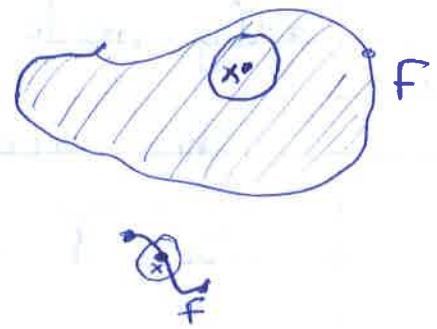
→ Observation: $F \subseteq \bar{F}$, for all $f \subseteq (X, d)$.

Proof: Let $x \in f$. Then, for any $r > 0$,

$$x \in f \cap B_d(x, r),$$

thus $f \cap B_d(x, r) \neq \emptyset$, $\forall r > 0$.
So, $x \in \bar{F}$.

Thus, $f \subseteq \bar{F}$.



→ Prop: For any $f \subseteq (X, d)$, \bar{F} is closed (w.r.t. d).

Proof: We will show that $X \setminus \bar{F}$ is open.

Let $x \in X \setminus \bar{F}$.

Since x is not in \bar{F} , there exists some open ball $B_d(x, r)$

that does not intersect f .

Thus, $B_d(x, r) \subseteq X \setminus \bar{F}$.

This implies that $B_d(x, r) \subseteq \underline{X \setminus \bar{F}}$: Let $y \in B_d(x, r)$.

smaller than
 $X \setminus \bar{F}$!



Since $B_d(x, r)$ is open, there exists an open ball $B_d(y, r_y)$ fully inside $B_d(x, r)$, thus inside $X \setminus \bar{F}$. So, $y \notin \bar{F}$, i.e. $y \in X \setminus \bar{F}$.

Therefore, we have shown that, for any $x \in X \setminus \bar{F}$, there exists an open ball centered at x , fully inside $X \setminus \bar{F}$.

This means that $X \setminus \bar{F}$ is open.

So, \bar{F} is closed.

\rightarrow Prop: Let (X, d) be a metric space, and $F \subseteq X$.

Then, $\bar{F} = \{x \in X : d(x, F) = 0\}$.



the set of
points in X at
distance 0 from F .

Proof: This will be an exercise. Note that

$$d(x, F) := \inf \{d(x, y) : y \in F\}.$$

the distance
of x from F is the "smallest"
distance between x and any element of F .

This gives us a good intuition about what the closure of a set is.

①

Lecture 35:

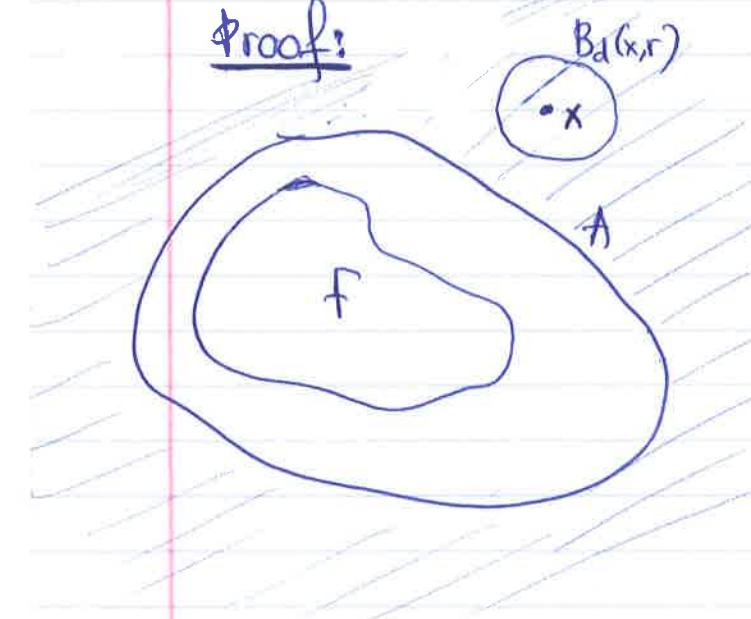
→ **Prop:** let (X, d) be a metric space.

for any $F \subseteq X$, \bar{F} is the smallest closed set containing F .

In other words: if $A \subseteq X$ is a closed set containing F , then $\bar{F} \subseteq A$.

(i.e., \bar{F} is smaller than all closed sets containing F).

Proof:



To show that $\bar{F} \subseteq A$, it suffices to show that

$$X \setminus A \subseteq X \setminus \bar{F}.$$

Let $x \in X \setminus A$ (\rightarrow we want to show that $x \notin \bar{F}$, i.e. x "far from" F).
 $X \setminus A$ is open, so there exists an open ball $B_d(x,r) \subseteq X \setminus A$.

$$\subseteq X \setminus \bar{F}.$$

as $\bar{F} \subseteq A$

②

That is, $B(x, r)$ doesn't intersect F . So, $x \notin F$,

i.e. $x \in X \setminus F$.

Since $x \in X \setminus A$ was arbitrary, we have: $X \setminus A \subseteq X \setminus F$.

So, $\bar{F} \subseteq A$.



Note that the above implies that.

$$F \text{ is closed} \Leftrightarrow F = \bar{F}$$

(as, if F is closed, then \bar{F} is the smallest closed set containing F).

Thus, understanding closures is understanding all closed sets.

We are now going to see a characterisation of closures:

(3)

→ Def: Let (X, d) be a metric space, and $A \subseteq X$.

Then, for any $x_0 \in X$, we say that x_0

is an accumulation point of A (w.r.t. d) if

we can find elements of A other than x_0 arbitrarily close to x_0 .

$$\forall r > 0, A \cap B_d(x_0, r) \setminus \{x_0\} \neq \emptyset$$

punctured neighbourhood of x_0 .



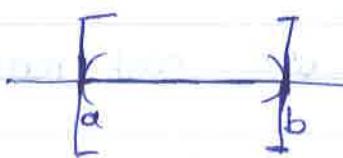
⚠️ Notice how, since the open balls depend on d , the accumulation points of A depend on d too.

→ Examples:

① In \mathbb{R} with the usual metric, the accumulation points are as we defined them long ago.

for instance, for any open interval (a, b) , with

$a, b \in \mathbb{R}$, the accumulation points of (a, b) are all the elements of $[a, b]$.



② In \mathbb{R}^2 , the open ball $B_d(x_0, r)$, for any $x_0 \in \mathbb{R}^2, r > 0$, is the open disc centered at x_0 , with radius r .

usual metric

(4)

The set of accumulation points of the open disc

$B_d(x_0, r)$ is the closed ball $\bar{B}_d(x_0, r)$



○

(which is the disc, together with its boundary).

circle of
radius r ,
with centre x_0 .

(3)

Let $X \neq \emptyset$ be a set, equipped with the discrete metric d . Then, for any $A \subseteq X$, A has no accumulation points. Indeed, no matter which $x \in X$ we pick, we can find a punctured neighbourhood of x_0 that contains no point of A ; in fact, no point at all:

$$B_d(x_0, \frac{1}{2}) \setminus \{x_0\} = \emptyset,$$

$$\text{thus } A \cap \left(B_d(x_0, \frac{1}{2}) \setminus \{x_0\} \right) = \emptyset;$$

so x_0 is not an accumulation point of A .

Mainly, all points are very far from each other w.r.t. this metric. So, A is very far, as a set, from all other points of X . While, by their definition, accumulation points of a set have to be very close to the set.

(arbitrarily, really).

(5)

→ Prop: Let (X, d) be a metric space, and $A \subseteq X$.

If $x_0 \in X$ is an accumulation point of A , then any punctured neighbourhood $B_d(x_0, r) \setminus \{x_0\}$ of x_0 contains infinitely many points of A .

Proof: Exercise, follows easily by the definition of an accumulation point.

This says, literally: the points of X arbitrarily close to A are the points of A , and the points arbitrarily close to A that may not be in A .

→ Prop. Let (X, d) be a metric space, and $A \subseteq X$.

Then,

$$\bar{A} = A \cup A'$$

the set
of accumulation
points of A .

Proof:

$$A \cup A' \subseteq \bar{A}$$

$A \subseteq \bar{A}$ (obvious)

and $A' \subseteq \bar{A}$: if $x \in A'$, then, $\forall r > 0$,

$$A \cap (B(x, r) \setminus \{x\}) \neq \emptyset, \text{ thus } B(x, r) \cap A \neq \emptyset \quad \forall r > 0.$$

So, $x \in \bar{A}$.

Idea: Whether $x \in A$ or $x \in A'$, it is arbitrarily close to A , so it is in the closure of A .

(6)

Idea: If $x \in \bar{A}$, then it is arbitrarily close to A . So, it is either in A , or not, in which case it is an accumulation point of A .

• $\bar{A} = A \cup A'$: Let $x \in \bar{A}$. If $x \in A$, then we are ok.

If $x \notin A$:



Since $x \in \bar{A}$, then $B(x, r) \cap A \neq \emptyset$, $\forall r > 0$.

i.e., we can find an element of A in $B(x, r)$

Since $x \notin A$, then $B(x, r) \cap A$ does not

contain x ; thus,
some $y \neq x$. In other words,

$$(B(x, r) \setminus \{x\}) \cap A \neq \emptyset, \forall r > 0.$$

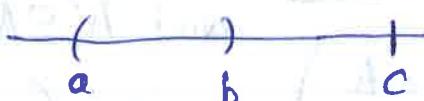
So, $x \in A'$.

→ Example: In \mathbb{R} with the usual metric, let $a < b$.

$$\text{Then, } \overline{(a, b)} = (a, b) \cup \underbrace{(a, b)'}_{= [a, b]} = [a, b].$$

(Note that this also implies that $[a, b]$ is closed, and that $[a, b]$ is the smallest closed set containing (a, b) !)

Let also $c \notin [a, b]$.



$$\begin{aligned} \text{Then, } \overline{(a, b) \cup \{c\}} &= ((a, b) \cup \{c\})' \cup ((a, b) \cup \{c\})' \\ &= (a, b) \cup \{c\} \cup [a, b] = [a, b] \cup \{c\}. \end{aligned}$$

(7)

→ Corollary: Let (X, d) be a metric space, and $F \subseteq X$.

Then, F is closed (w.r.t. d)



$$F = \bar{F} = F \cup F'$$

i.e. $\Leftrightarrow F$ contains its accumulation points



Note that $F \supseteq F'$ doesn't mean that F has accumulation points; it could be that $F' = \emptyset$.

→ Sequences in metric spaces:

→ Def: A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) is a function $x: \mathbb{N} \rightarrow X$,
with $x(n) = x_n, \forall n \in \mathbb{N}$.

→ Def: Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in the metric space (X, d) .
We say that $(x_n)_{n \in \mathbb{N}}$ converges to $x_0 \in X$,
and we write $x_n \xrightarrow{d} x_0$, or $x_n \rightarrow x_0$,

if: $\forall \varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ st.:
 $\forall n > n_0, x_n \in B_d(x_0, \varepsilon)$.

This is the same as saying that:

$\forall \varepsilon > 0$, there exists a whole final part of $(x_n)_{n \in \mathbb{N}}$
fully contained in $B_d(x_0, \varepsilon)$.

$$(x_1, x_2, \dots, \boxed{x_{n_0}, x_{n_0+1}, \dots})$$



Also, it is the same as saying that:

(9)

Also, there exists $n_0 \in \mathbb{N}$ s.t. :

$$\forall n \geq n_0, \quad d(x_n, x_0) < \varepsilon.$$

i.e. $\underbrace{d(x_n, x_0)}_{\in \mathbb{R}} \rightarrow 0$

i.e. :

$$x_n \xrightarrow{d} x_0 \iff d(x_n, x_0) \xrightarrow{n \rightarrow +\infty} 0.$$



Notice that, in a normed space $(X, \|\cdot\|)$,

a sequence $(x_n)_{n \in \mathbb{N}}$ in X converges to $x \in X$

$$\iff \|x_n - x_0\| \xrightarrow{n \rightarrow +\infty} 0.$$

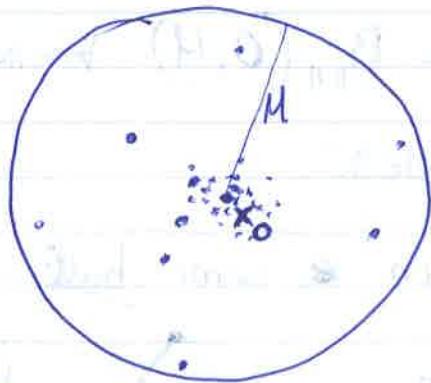
We tend to denote this by $x_n \xrightarrow{\|\cdot\|} x_0$ (but it really is $x_n \xrightarrow{d(X, \|\cdot\|)} x_0$)

The basic properties of limits of sequences that we proved in \mathbb{R} similarly follow for sequences in metric (and normed) spaces:

→ Properties of sequences in metric spaces:

Let (X, d) be a metric space. Then:

- ① If a sequence $(x_n)_{n \in \mathbb{N}}$ in X converges in X , then its limit is unique.
- ② If a sequence $(x_n)_{n \in \mathbb{N}}$ in X converges in X , then every subsequence $(x_{k_n})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ has the same limit.
- ③ If a sequence $(x_n)_{n \in \mathbb{N}}$ in X converges to $x_0 \in X$, then it is bounded; meaning that there exists $N > 0$ st. $x_n \in B_d(x_0, N)$, $\forall n \in \mathbb{N}$.



Note that X is not necessarily a vector space, so we can't necessarily be talking about a ball centered at 0 , as we can't necessarily be talking about a 0 !

Similarly, we can't necessarily be talking about sums of sequences, as addition may not be defined on X ...

(11)

Extra properties of sequences in normed spaces:

Let $(X, \|\cdot\|)$ be a normed space.

Let $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ be sequences in X ,

with $x_n \xrightarrow{\|\cdot\|} x_0 \in X$ and $y_n \rightarrow y_0 \in X$.

(i.e. $\lim_{n \rightarrow \infty} \|x_n - x_0\| \rightarrow 0$)
(convergence w.r.t. the metric $d_{\|\cdot\|}$)

(i.e. $\lim_{n \rightarrow \infty} \|y_n - y_0\| \rightarrow 0$)

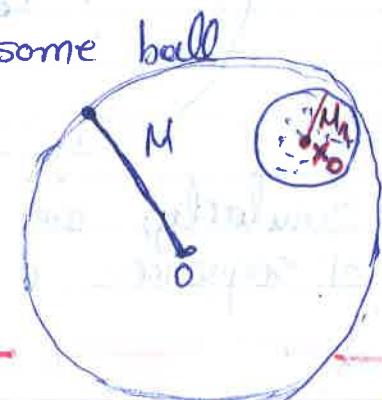
Then:

① $\underbrace{x_n + y_n}_{\text{well-defined}} \rightarrow x_0 + y_0$ ($\in X$, as X is a vector space).

② $\underbrace{\lambda x_n}_{\text{well-defined}} \rightarrow \lambda x_0$ ($\in X$, as X is a vector space),
 for all scalars $\lambda \in \mathbb{R}$.

③ There exists $M > 0$, $x_n \in B_{\|\cdot\|}(0, M)$ $\forall n \in \mathbb{N}$,
 i.e. s.t. $\|x_n\| < M$, $\forall n \in \mathbb{N}$.

(we know that all x_n 's are in some ball centered at x_0 ; and, in turn, this ball is contained in some ball centered at 0).



Prop:

Characterisation of closures via sequences:

Let (X, d) be a metric space, and $A \subseteq X$.

Then, $\bar{A} = \left\{ \text{limits}^{\text{w.r.t. } d} \text{ of all sequences in } A \text{ that converge in } X \right\}$

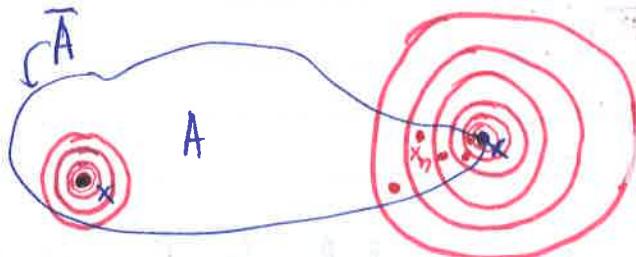
I.e., $\forall x \in X$, we have:

$x \in \bar{A} \iff \text{there exists a sequence } (x_n)_{n \in \mathbb{N}} \text{ in } A \text{ (i.e., with } x_n \in A \text{ for all } n \text{), such that } x_n \xrightarrow{d} x.$

Proof: (\implies) Let $x \in \bar{A}$. We will construct a sequence in A that converges to x .

(Idea: $x \in \bar{A}$ means that A gets arbitrarily close to x , so the above must be easy.)

i.e., \exists element of A at distance $< \frac{1}{n}$ from x .



Since $x \in \bar{A}$, $\forall n \in \mathbb{N}$, $B_d(x, \frac{1}{n}) \cap A \neq \emptyset$,

$\exists x_n \in A$, with $d(x_n, x) < \frac{1}{n}$

(note that, for $x \in \bar{A}$, we can be choosing)
 $x_n = x \quad \forall n \in \mathbb{N} \dots$

(13)

$$\text{So, } 0 \leq d(x_n, x) < \frac{1}{n} \quad \forall n \in \mathbb{N},$$

$\downarrow n \rightarrow \infty$ $\downarrow n \rightarrow \infty$

0 0

thus $d(x_n, x) \rightarrow 0$ (the distance between x_n and x gets smaller and smaller),
 i.e. $x_n \xrightarrow{d} x$.

Thus, $(x_n)_{n \in \mathbb{N}}$ is a sequence in A with the property we have been looking for.

\leftarrow We know that there exists $(x_n)_{n \in \mathbb{N}}$ in A , with $x_n \xrightarrow{d} x$. We want: $x \in \bar{A}$

Idea: Since the x_n 's are in A , and they get arbitrarily close to x , A is getting arbitrarily close to x , $\Rightarrow x \in \bar{A}$.

Let $r > 0$.



Since $x_n \xrightarrow{d} x$, there exists a whole final part of $(x_n)_{n \in \mathbb{N}}$ inside $B_r(x)$. Thus, $B_r(x) \cap A \neq \emptyset$. (as $x_n \in A \quad \forall n \in \mathbb{N}$). So, $x \in \bar{A}$ ($\because r > 0$ was arbitrary).

→ Corollary: Characterisation of closed sets via sequences:

Let (X, d) be a metric space, and $f \subseteq X$.

Then, f is closed



$f = \bar{f} = \{ \text{limits w.r.t. } d \text{ of all sequences in } f \text{ that converge in } X \}$



It is **vital** here that we check whether the limit of any sequence $(x_n)_{n \in \mathbb{N}}$ in f is in f , when

$(x_n)_{n \in \mathbb{N}}$ converges to a point in \bar{X} (rather than outside our "world" X).

for instance: for $X = (0, 1)$, equipped with the usual metric d , X is closed (we know this is true for any topology on X).

This implies that, whenever we look at a sequence $(x_n)_{n \in \mathbb{N}}$ in X , with $x_n \xrightarrow{d} x \in \underline{X}$, then $x \in X$. This obviously holds.

The fact that $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ is a sequence in X , and $\frac{1}{n} \xrightarrow{d} 0 \notin X$, doesn't mean that X is not closed:

(15)

this $(\frac{1}{n})_{n \in \mathbb{N}}$ is not a sequence that converges in our "world" X , so its limit is not required to be in X for X to be closed. (technically for us it is not convergent, because it doesn't converge in our "world" X)

The above provides the following techniques:

Let (X, d) be a metric space, and $F \subseteq X$:

→ Technique to show that F is closed: w.r.t. d

Consider the arbitrary $(x_n)_{n \in \mathbb{N}}$ in F ,

with $x_n \xrightarrow{d} x \in X$ (for some $x \in X$).

Show that $x \in F$.

(Here, the knowledge that $(x_n)_{n \in \mathbb{N}}$ converges, and its limit is in X , gives some information that hopefully helps us to deduce that $x \in F$.)

→ Technique to show that F is not closed:

find a sequence $(x_n)_{n \in \mathbb{N}}$ in F , with

$(x_n)_{n \in \mathbb{N}}$ converging (w.r.t. d) to some $x \in X \setminus F$.

Lecture 36:

①

Cauchy sequences in metric spaces:

We define Cauchy sequences in this general setting just like in \mathbb{R} :

We don't even need X for this definition... All we need is a metric d on $\{(x_n)_{n \in \mathbb{N}}\}$.

→ Def: Let (X, d) be a metric space. A sequence

⚠ This defn. is independent of the other elements of X , i.e. of the ambient space in which $(x_n)_{n \in \mathbb{N}}$ lives!

$(x_n)_{n \in \mathbb{N}}$ in X is called a Cauchy sequence w.r.t. d

if: $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n, m \geq n_0, d(x_n, x_m) < \varepsilon$.

→ Prop: Let (X, d) be a metric space, and $(x_n)_{n \in \mathbb{N}}$ a sequence in X . Then:

(i) If $(x_n)_{n \in \mathbb{N}}$ converges in X , then $(x_n)_{n \in \mathbb{N}}$ is Cauchy.

(ii) The converse of (i) is not always true

(i.e., Cauchy sequences in X don't necessarily converge inside X).

Proof: (i) As in \mathbb{R} :

$(x_n)_{n \in \mathbb{N}}$ converges
in X ; thus, $\exists x \in X$ s.t. $x_n \xrightarrow{d} x$.

Let $\varepsilon > 0$. Since $x_n \xrightarrow{d} x$, $\exists n_0 \in \mathbb{N}$ st.:

$$\forall n \geq n_0, d(x_n, x) < \frac{\varepsilon}{2}.$$

$$\text{So, } \forall n, m \geq n_0 : d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) \underset{\text{triangle inequality}}{\leq} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(2)

Since $\epsilon > 0$ was arbitrary, $(x_n)_{n \in \mathbb{N}}$ is Cauchy.

(ii) Let $X = (0, +\infty)$, equipped with the usual metric d on \mathbb{R} . Consider the sequence $(\frac{1}{n})_{n \in \mathbb{N}}$ in X .

We know that, in (\mathbb{R}, d) , $\frac{1}{n} \xrightarrow{d} 0$.

So, since $(\frac{1}{n})_{n \in \mathbb{N}}$ converges in (\mathbb{R}, d) , it is Cauchy w.r.t. d .

However, $0 \notin X$, thus $(\frac{1}{n})_{n \in \mathbb{N}}$ doesn't converge in X (if it did, it would have two distinct limits in (\mathbb{R}, d) , contradiction).



Def: Let (X, d) be a metric space. We say that

(X, d) is a **complete metric space**

if

every Cauchy sequence in (X, d) converges in X .

w.r.t. d

w.r.t. d

→ Examples: ① $(\mathbb{R}, \text{usual metric})$ is complete
(this followed by the Bolzano-Weierstrass thm.).

! → Think of this as:
X contains the limits of its Cauchy sequences!

(3)

(2) $(0, +\infty)$, equipped with the usual metric of \mathbb{R} , is not complete (proved above).

This says: complete is a special case of closed.

Prop: Let (X, d) be a metric space, and $Y \subseteq X$.

If (Y, d) is complete, then Y is closed in (X, d) .

really, the restriction of d on Y

[Here, we see a property of (Y, d) , that only depends on (Y, d) , implying things about the completion of Y with larger spaces with extended metrics]

Proof: Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in Y ,
s.t. $x_n \xrightarrow{d} x$, for some $x \in X$.

We show that $x \in Y$:

[This implication makes things independent of the ambient space]

$(x_n)_{n \in \mathbb{N}}$ converges in $X \Rightarrow (x_n)_{n \in \mathbb{N}}$ Cauchy w.r.t. d

$\xrightarrow{(x_n)_{n \in \mathbb{N}} \text{ in } Y, (Y, d) \text{ complete}}$ $(x_n)_{n \in \mathbb{N}}$ converges in Y . Thus, $x \in Y$.

Since $(x_n)_{n \in \mathbb{N}}$ with the above property was arbitrary,
 Y is closed in (X, d) .

→ Example: $(0, +\infty)$, equipped with the usual metric of \mathbb{R} , is not closed in \mathbb{R} : $\frac{1}{n} \in (0, +\infty)$ for $n \in \mathbb{N}$, and $\frac{1}{n} \rightarrow 0 \notin (0, +\infty)$.

So, $(0, +\infty)$ is not complete with this metric.

[this is really just repeating our observations on the definition of Cauchy sequences, regarding independence on the ambient space.]

(4)

⚠ In the above proof, we made the observation that, in a metric space (X, d) , if $(x_n)_{n \in \mathbb{N}}$ is Cauchy, then it is also Cauchy in any (Y, d) , $\forall Y \subseteq X$, s.t. $(x_n)_{n \in \mathbb{N}}$ lives in Y .

(While of course $(x_n)_{n \in \mathbb{N}}$ may converge to some $x \in X$ with $x \notin Y$).

So, the notion of a Cauchy sequence is independent of the space containing $(x_n)_{n \in \mathbb{N}}$ (it only depends on the distances between the elements of the sequence),

unlike the notion of convergent sequence.

That is why it is useful many times to think about Cauchy sequences; we know we have "closeness" between the terms of the sequence, without caring about where the limit is, or what the ambient space looks like.

(5)



If you are interested: for every metric space (X, d) , there exists a larger metric space (\tilde{X}, \tilde{d}) (with $\tilde{X} \supseteq X$, \tilde{d} an extension of d), such that (\tilde{X}, \tilde{d}) is complete.

The smallest (\tilde{X}, \tilde{d}) with this property is called the completion of (X, d) : it is the smallest metric space in which all Cauchy sequences in (X, d) converge!

The above implies that, if $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in (X, d) , then certainly $(x_n)_{n \in \mathbb{N}}$ has a limit in some larger space! Thus, the only obstruction

to (X, d) being complete is that it may not contain the limit points of its Cauchy sequences

(i.e. that it may not be closed in its completion.),
(or, really, equal to its completion...)

(Not that it contains Cauchy sequences that don't have limit points in any larger universe.)

Hopefully this makes the following proposition intuitive:

Inside a complete space, complete is the same as closed. ⑥

Prop: Let (X, d) be a complete metric space.

Let $Y \subseteq X$. We consider the metric space (Y, d)

the restriction
of d on Y .

Then, (Y, d) is complete $\Leftrightarrow Y$ closed in (X, d) .

Proof: (\Rightarrow) follows by previous Proposition
(whether or not (X, d) is complete).

(\Leftarrow) Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in (Y, d) .

We will show that $(x_n)_{n \in \mathbb{N}}$ converges in Y :

$(x_n)_{n \in \mathbb{N}}$ Cauchy w.r.t. $d \rightarrow (x_n)_{n \in \mathbb{N}}$ Cauchy in (X, d)

$\overbrace{(X, d)}$ complete $\overbrace{(x_n)_{n \in \mathbb{N}}}$ converges to some $x \in X$
(w.r.t. d)

Since $(x_n)_{n \in \mathbb{N}}$ is a sequence in Y , and Y closed
in (X, d) , we have that $x \in Y$.

Thus, $(x_n)_{n \in \mathbb{N}}$ converges in Y . ■

(7)

Complete metric spaces are very important on their own right. Here, we will use them to understand better compact sets in metric spaces.

Compact sets depend on the topology we have imposed on our space. Changing the topology can make it easier for sets to be compact, and thus it can become ~~possible~~ for us to use the theory behind compact to understand properties of our space.

→ Compact sets:

→ Def: Let (X, d) be a metric space, and $K \subseteq X$.

We say that a family \mathcal{U} of subsets of X is an open cover of K in (X, d) if:

- (i) each $U \in \mathcal{U}$ is an open subset of (X, d) !
- (ii) $K \subseteq \bigcup_{U \in \mathcal{U}} U$

We say that a family $\mathcal{U}' \subseteq \mathcal{U}$ is a subcover of \mathcal{U}

if also $K \subseteq \bigcup_{U \in \mathcal{U}'} U$

(i.e., if K is covered also by the union of the fewer sets in \mathcal{U}').

→ This definition generalises to any topological space (X, τ) !
(See how no metric is needed for this definition to make sense; only open sets are needed.) (8)

! Def: Let (X, d) be a metric space.

We say that (X, d) is **compact** if

every open cover of X (in (X, d)) has a finite subcover.

I.e., if: for any family \mathcal{A} of open subsets of (X, d) ,
with $X = \bigcup_{U \in \mathcal{A}} U$,

there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$, of finitely many sets, s.t. $X = \bigcup_{U \in \mathcal{A}'} U$

(i.e., for any open sets that cover X , we can find finitely many of them that also cover X).
as a union!

! Def: Let (X, d) be a metric space, and $K \subseteq X$.

We say that K is a **compact subset** of X

if (K, d) is a compact metric space.
the restriction of d on K .

Remember, the open subsets of (K, d) are the sets of the form $\bigcup_{U \in \mathcal{A}} K$, where U is open in (X, d) .
(Weekly Assignment 13).

(9)

So, K is a compact subset of (X, d)



every open cover of K in (X, d)

has a finite subcover;

i.e., \iff for any family \mathcal{U} of open subsets of (X, d) , with $K \subseteq \bigcup_{U \in \mathcal{U}} U$,

$$\text{as } K = \bigcup_{U \in \mathcal{U}} (U \cap K) \hookrightarrow K \subseteq \bigcup_{U \in \mathcal{U}} U$$

there exists a subfamily $\mathcal{U}' \subseteq \mathcal{U}$, of finitely many sets, s.t.

$$K \subseteq \bigcup_{U \in \mathcal{U}'} U.$$

(i.e., for any open sets in (X, d) , that cover K)

there exist finitely many of them that cover K .)



Examples:

① Let (X, d) be a metric space. Then, any finite subset of X is compact.

Proof: Let $K = \{x_1, \dots, x_n\} \subseteq X$. Let \mathcal{U} be an open cover of K in X .

(10)

Then, we can certainly find finitely many sets in \mathcal{U} whose union covers K :

$$\{x_1, \dots, x_n\} \subseteq \bigcup_{i=1}^n U_i, \text{ so, } \forall i=1, \dots, n,$$

there exists U_i s.t. $x_i \in U_i$.

$$\text{Thus, } \{x_1, \dots, x_n\} \subseteq \bigcup_{i=1}^n U_i.$$

So, $\{U_1, \dots, U_n\}$ is a finite subcover of $\{x_1, \dots, x_n\}$ in \mathcal{U} .

(2) \mathbb{R} , equipped with the usual metric, is not compact.:

$$\text{We have } \mathbb{R} = \bigcup_{n=1}^{+\infty} (-n, n).$$

open subsets
of \mathbb{R} w.r.t.
the usual
metric

So, $\mathcal{A} = \{(-n, n) : n \in \mathbb{N}\}$ is an open cover of \mathbb{R} .

But it doesn't have a finite subcover:

Indeed, suppose that it does, i.e. that there exist finitely many $n_1 < n_2 < \dots < n_k$ in \mathbb{N} ,

$$\text{with } \mathbb{R} = \bigcup_{i=1}^k (-n_i, n_i) = \overbrace{(-n_1, n_1)}^{(-n_1, n_1) \subseteq (-n_2, n_2) \subseteq \dots \subseteq (-n_k, n_k)} \subseteq (-n_k, n_k).$$

Then, $\mathbb{R} = (-n_k, n_k)$, a contradiction.

③ Let (X, d) be a metric space.

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X , that converges in X .

Then, $\{x_n : n \in \mathbb{N}\}$ is compact. (Exercise.).



One of our final goals is to show that, for any metric space (X, d) ,

(X, d) is compact \iff the Bolzano-Weierstrass theorem holds in (X, d)

[i.e., \iff any sequence in (X, d)
has a convergent subsequence
(convergent in X)]

In particular, closed intervals $[a, b]$ in \mathbb{R} , with the usual metric, are compact.

And compact sets in general have many of the nice properties of closed intervals

(e.g., continuous functions send compact sets to compact sets, Bolzano-Weierstrass, etc.).

(12)

Prop: Let (X, d) be a metric space, and let $K \subseteq X$

Then, K compact $\rightarrow K$ closed and bounded w.r.t. d in (X, d)

But actually converse true
in $(\mathbb{R}^n, \text{eucl metric})$
will see this later

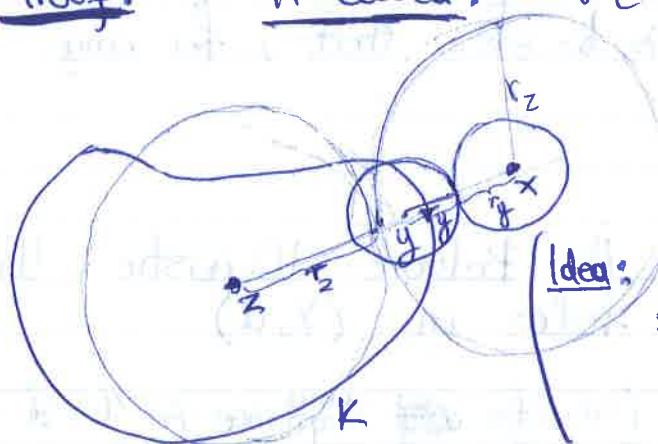
⚠ The converse is NOT true!

Example: $(\mathbb{R}, \text{discrete metric})$ is not compact, } Exercise
but it is closed and bounded. 'in itself'

Proof: • K closed: We will show that $X \setminus K$ is open.

let $x \in X \setminus K$.

We want to find $r > 0$ s.t.
 $B_d(x, r) \subseteq X \setminus K$



Idea: For each $y \in K$, we can find a ball centered at y , at some distance from x :

or smaller.

Hence, let $r_y = \frac{1}{2} d(x, y) > 0$

Then, $B_d(x, r_y) \cap B_d(y, r_y) = \emptyset$

Now, $K \subseteq \bigcup_{y \in K} B_d(y, r_y)$ open in (X, d) i.e., $\{B_d(y, r_y) : y \in K\}$ is an open cover of (X, d)

Since K is compact in (X, d) , the above implies

that we can find finitely many $y_1, \dots, y_n \in K$,

s.t. $K \subseteq \bigcup_{i=1}^n B_d(y_i, r_{y_i})$

Now, $B_d(x, r_{y_i})$ doesn't intersect $B_d(y_i, r_{y_i})$,

$$\forall i = 1, \dots, n$$

$\Rightarrow \bigcap_{i=1}^n B_d(x, r_{y_i})$ doesn't intersect $B_d(y_i, r_{y_i})$
 smaller than each $B_d(x, r_{y_i})$ for any $i = 1, \dots, n$;

it thus doesn't intersect
 $\bigcup_{i=1}^n B_d(y_i, r_{y_i}) \supseteq K$,

thus $\left(\bigcap_{i=1}^n B_d(x, r_{y_i}) \right) \cap K = \emptyset$,

i.e. $\bigcap_{i=1}^n B_d(x, r_{y_i}) \subseteq X \setminus K$

Now: $\bigcap_{i=1}^n B_d(x, r_{y_i}) = B_d(x, r)$, where $r = \min \{r_{y_1}, \dots, r_{y_n}\} > 0$

that is why it was so important to consider a finite subcover; it made these concentric balls finitely many, thus their intersection

is non-empty! Without that, we

would only manage to get $\bigcap_{y \in K} B_d(x, r_y) \subseteq X \setminus K$

$\underbrace{\quad}_{\text{may well be empty,}}$

so useless.

Thus, for this $r > 0$, $B_d(x, r) \subseteq X \setminus K$. Since $x \notin X \setminus K$ was arbitrary, $X \setminus K$ is open in $(X, d) \Rightarrow K$ is closed.

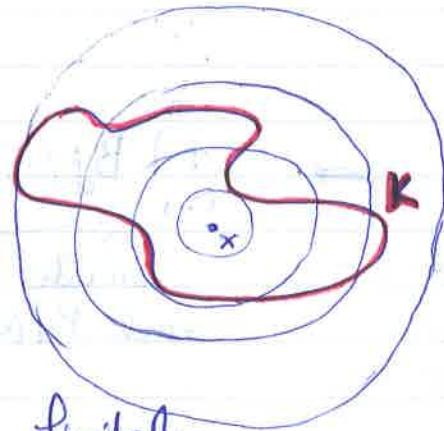
- K is bounded: We want to show that K is contained in some ball in (X, d) .

Fix some $x_0 \in K$ (doesn't matter which)

We have:

$$K \subseteq \bigcup_{r>0} B_d(x_0, r).$$

open



Since K is compact, there exist finitely

many $r_1, \dots, r_n > 0$, st. $K \subseteq \bigcup_{i=1}^n B_d(x_0, r_i) =$
union of
finitely many
concentric balls;
a ball!

$$= B_d(x_0, r), \text{ for } r = \max\{r_1, \dots, r_n\}.$$

I.e., $K \subseteq B_d(x_0, r)$. So, K is bounded.

Lecture 37:

(1)

→ Continuous functions on metric spaces :

These are defined in the same way as continuous functions from \mathbb{R} to \mathbb{R} . However, we are now in a position to show an equivalence of the definition with a property that is purely topological (i.e., that makes sense in spaces with a topology, whether this comes from a metric or not.).

The reason we are interpolating this in our study of compact sets is that continuous functions interact very well with compact sets (this therefore gives us a motivation to study compact sets further).

→ Def: Let (X, d_X) , (Y, d_Y) be metric spaces.

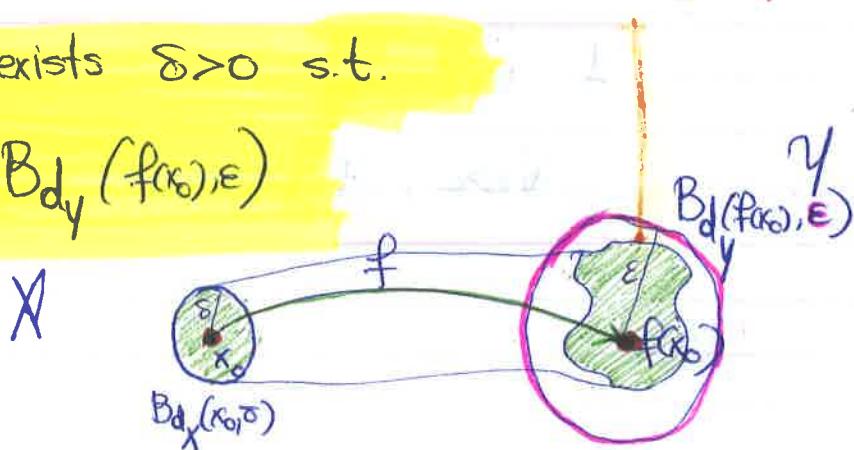
Let $f: X \rightarrow Y$. We say that

f is continuous at $x_0 \in X$ (w.r.t. these metrics)

if: $\forall \epsilon > 0$, there exists $\delta > 0$ s.t.

$$f\left(B_{d_X}(x_0, \delta)\right) \subseteq B_{d_Y}(f(x_0), \epsilon)$$

i.e., there exists some neighbourhood of x_0 that is fully contained in the neighbourhood of $f(x_0)$.



(2)

Just like for $f: \mathbb{R} \rightarrow \mathbb{R}$ (with the usual metric on both domain and target space),

this means that, no matter how close we look at around $f(x_0)$

(i.e., no matter what ball $B_{d_y}(f(x_0), \epsilon)$ we fix),

we will find there the images of all $x \in X$ that are sufficiently close to x_0 w.r.t. d_X

depends
on ϵ, x_0

(i.e., a whole ball $B_{d_Y}(x_0, \delta)$ is mapped inside $B_{d_Y}(f(x_0), \epsilon)$).

i.e., $f(x)$ is close to $f(x_0)$ when x gets close enough to x_0 w.r.t. d_X

See that the above can be rephrased as:

f continuous at x_0 if:

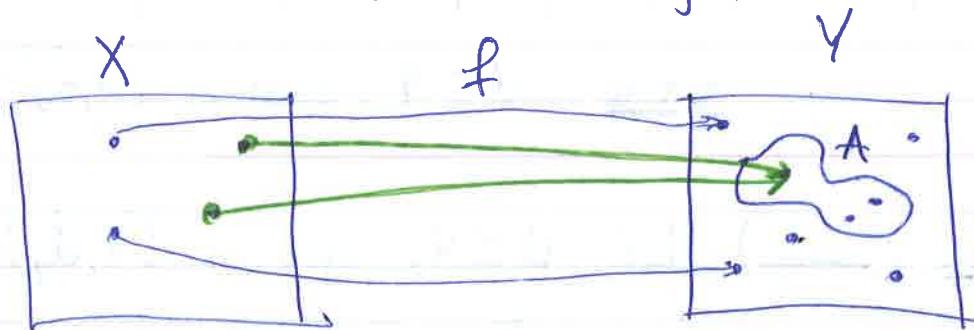
$\forall \epsilon > 0, \exists \delta > 0$ s.t. : if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \epsilon$

Now, for any $A \subseteq Y$, we define

$$f^{-1}(A) := \{x \in X : f(x) \in A\}$$

the inverse image of A

i.e., $f^{-1}(A)$ consists of all the elements of X that are sent inside A via f :



Note that f doesn't have to be 1-1 and onto for $f^{-1}(A)$ to be defined (just as in the picture above).

I.e., f^{-1} doesn't have to be a well-defined function;

$f^{-1}(A)$ is well-defined if $A \subseteq Y$, whether

the function $f^{-1}: Y \rightarrow X$ is well-defined or not.



$B \subseteq f^{-1}(A) \iff B$ is sent inside A via f
 $\iff f(B) \subseteq A$.

! Very important!

(4)

Thm: Let (X, d_X) , (Y, d_Y) be metric spaces,
and let $f: X \rightarrow Y$.

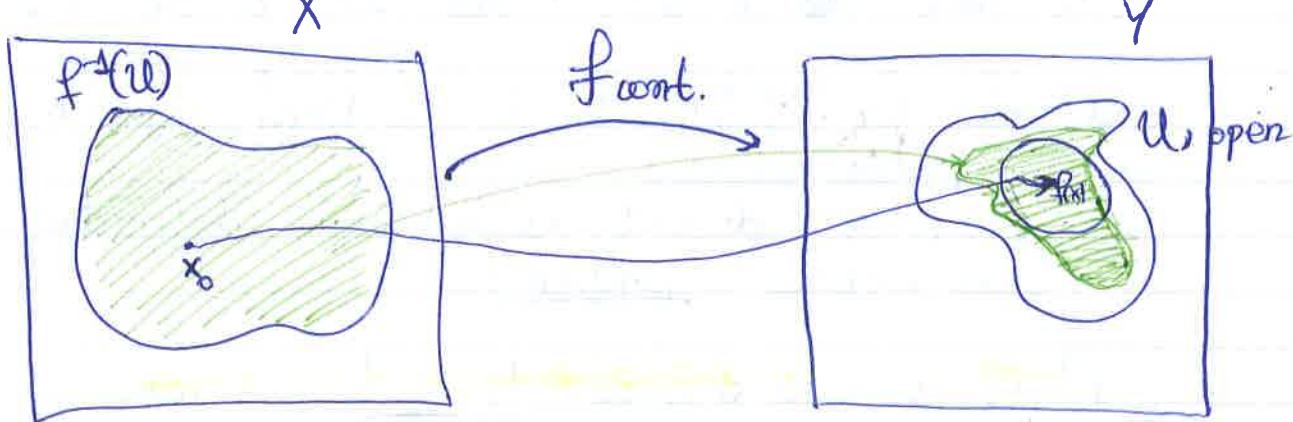
Then, f continuous (w.r.t. these metrics)

\Leftrightarrow
 f inverts open sets to open sets.

(i.e., $f^{-1}(U)$ open in (X, d_X) , for
every $U \subseteq Y$, open in (Y, d_Y) .)

Proof: (\Rightarrow) Let $U \subseteq Y$, open in (Y, d_Y) .

We want to show that $f^{-1}(U)$ open in (X, d_X)



Let $x_0 \in f^{-1}(U)$. We will show that there exists some open ball in (X, d_X) , centered at x_0 , fully inside $f^{-1}(U)$:

(5)

Indeed:

U is open, so there exists some $\epsilon > 0$, s.t.

$$B_{d_Y}(f(x_0), \epsilon) \subseteq U.$$

Since f is continuous at x_0 , for this $\epsilon > 0$ we can find a $\delta > 0$ s.t.

$$\underbrace{f(B_{d_X}(x_0, \delta)) \subseteq B_{d_Y}(f(x_0), \epsilon)} \subseteq U.$$

i.e., there exists some ball centered at x_0 , that is fully sent inside $B_{d_Y}(f(x_0), \epsilon)$ via f .

Thus, $B_{d_X}(x_0, \delta) \subseteq f^{-1}(U)$

Since $x_0 \in f^{-1}(U)$ was arbitrary, $f^{-1}(U)$ is open in (X, d_X)

(\Leftarrow) Let $x_0 \in X$. We will show that f is continuous at x_0 .

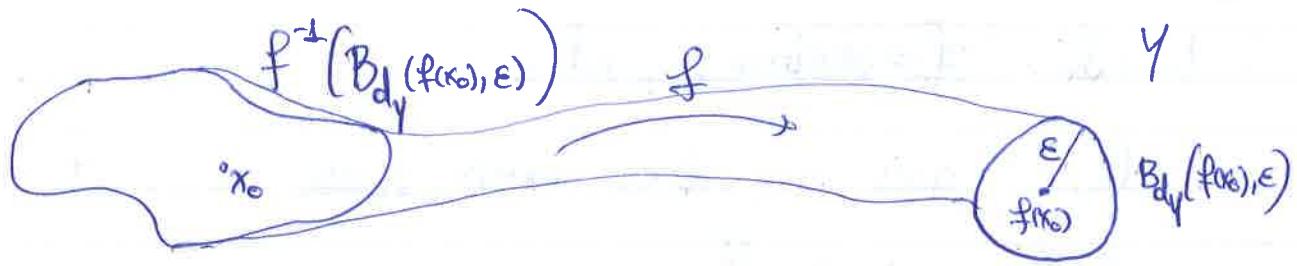
Indeed, let $\epsilon > 0$. We want to show that $\exists \delta > 0$

$$\text{s.t. } f(B_{d_X}(x_0, \delta)) \subseteq B_{d_Y}(f(x_0), \epsilon),$$

(i.e. s.t. $B_{d_X}(x_0, \delta) \subseteq f^{-1}(B_{d_Y}(f(x_0), \epsilon))$).

Since $B_{d_Y}(f(x_0), \epsilon)$ is open in (Y, d_Y) , and f inverts open sets to open sets, $f^{-1}(B_{d_Y}(f(x_0), \epsilon))$ is open in (X, d_X) .

(6)



Now, f inverts $f(x_0)$ to x_0 ;

thus, $x_0 \in f^{-1}(B_{d_Y}(f(x_0), \epsilon))$
 open in (X, d)

So, $\exists \delta > 0$ s.t. $B_{d_X}(x_0, \delta) \subseteq f^{-1}(B_{d_Y}(f(x_0), \epsilon))$
 $\rightarrow f(B_{d_X}(x_0, \delta)) \subseteq B_{d_Y}(f(x_0), \epsilon)$

Since x_0 was arbitrary, f is cont. at x_0 .

Since $x_0 \in X$ was arbitrary, f is cont. everywhere in X .

A See how f inverting open sets to open sets

doesn't require the existence of metrics on X and Y to make sense; it only requires the existence of topologies on X and Y !

Thus, this is how continuous functions are defined on topological spaces: $f: (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ is called continuous if f inverts open sets to open sets!
 i.e., if $f^{-1}(U) \in \mathcal{T}_Y$, $\forall U \in \mathcal{T}_X$.

(7)

The following nice interaction between continuous functions is actually valid on topological spaces too, and the proof is identical (no metric is required to define compact sets, remember).

→ Thm: Continuous functions send compact sets to compact sets.

I.e.: Let (X, d_X) , (Y, d_Y) be metric spaces
(or, even better, (X, τ_X) , (Y, τ_Y) topological spaces in general)

Let $f: X \rightarrow Y$ be continuous w.r.t. the metrics
(or, the topologies) on X and Y .

Then, $f(K)$ compact subset of (Y, d_Y) ,
(or, of (Y, τ_Y))

for any K compact subset of (X, d_X)
(or, of (X, τ_X)).

See that d_X, d_Y only feature by inducing topologies on X and Y , respectively. The proof would have worked just fine for any other topologies though.

Proof: Let K be a compact subset of (X, d_X) .

We will show that $f(K)$ is compact in (Y, d_Y) :

Let \mathcal{U} be an open cover of $f(K)$ in (Y, d_Y) .
We will show that \mathcal{U} has a finite subcover:

Indeed,

$$f(K) \subseteq \bigcup_{U \in \mathcal{U}}$$

remember for later: U open in (Y, d_Y) $\Leftrightarrow f^{-1}(U)$ open in (X, d_X)

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_{U \in \mathcal{U}} U\right) =$$

$$= \bigcup_{U \in \mathcal{U}} (f^{-1}(U))$$

open in (X, d_X)
 f cont., so inverts
open sets to open sets.

actually,
we have equality:
the points of X
that are sent in $f(K)$ via f
are exactly the elements
of K ... But this
is not important here...

Whole idea:

Since the red sets cover K , those that are inverted to them cover $f(K)$

K compact
in (X, d_X) ,

so this open cover
of K has a finite
subcover

$$\exists U_1, \dots, U_n \in \mathcal{U} \text{ s.t. } K \subseteq \bigcup_{i=1}^n f^{-1}(U_i) = f^{-1}\left(\bigcup_{i=1}^n U_i\right),$$

$$\text{so } f(K) \subseteq \bigcup_{i=1}^n U_i.$$

Thus, $\{U_1, \dots, U_n\}$ is a finite subcover of \mathcal{U} .

(covering $f(K)$).

Since \mathcal{U} was an arbitrary open cover of $f(K)$
in (Y, d_Y) , $f(K)$ is compact in (Y, d_Y) .

We now see some definitions, which will build up to the big theorem we will gradually prove.

Def:

Let (X, d) be a metric space.

We say that (X, d) is sequentially compact

(or, that the Bolzano-Weierstrass property holds in (X, d))

if :

every sequence in X has a subsequence that converges in X (w.r.t. d)



Just as for compactness:

If (X, d) is a metric space, and $K \subseteq X$,

we say that K is sequentially compact (in (X, d) , usually implied)

if (K, d) is sequentially compact

(i.e., if : every sequence in K has a

subsequence that converges in K w.r.t. d).

→ Examples:

1 In (\mathbb{R}, d) , where d is the usual metric, any interval

$[a, b]$, with $a \leq b$ in \mathbb{R} , is sequentially compact:
(i.e., $([a, b], d)$ is seq. compact)

Proof: Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $[a, b]$.

So, $(x_n)_{n \in \mathbb{N}}$ is bounded in \mathbb{R} .

Thus, by the Bolzano-Weierstrass theorem,

there exists a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$,

with $x_{k_n} \xrightarrow{d} x$, for some $x \in \mathbb{R}$.

Now, $x \in [a, b]$:

$$\boxed{\begin{array}{l} a \leq x_{k_n} \leq b \quad \forall n \in \mathbb{N} \\ \Rightarrow a \leq \lim_{\substack{n \rightarrow \infty \\ \parallel}} x_{k_n} \leq b \Rightarrow x \in [a, b] : \\ \text{our old proof.} \end{array}}$$

There is a faster proof with our new knowledge:

18

$x \in [a, b]$ because $[a, b]$ is closed in (\mathbb{R}, d) . ■

2 In (\mathbb{R}, d) , where d is the usual metric,

(a, b) is not sequentially compact, for any

$a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$: find a sequence $(x_n)_{n \in \mathbb{N}}$

in (a, b) that converges to a (when $a \in \mathbb{R}$),

or that decreases to $-\infty$ (when $a = -\infty$).

Any subsequence then doesn't converge inside (a, b) .

3 Similarly, in any metric space (X, d) , and

for any $F \subseteq X$: if F is not closed in (X, d) ,

then it is not sequentially compact:

Proof: Since F is not closed in (X, d) ,

we can find $(x_n)_{n \in \mathbb{N}}$ in F , with $x_n \xrightarrow{d} x \in X \setminus F$.

Thus, every subsequence of $(x_n)_{n \in \mathbb{N}}$ converges

to x , i.e. outside F .

11

③ means that if $f \subseteq (X, d)$ is sequentially compact
then it is closed in (X, d) .

We will eventually show something better:

compact \longleftrightarrow sequentially compact



③ means that if $f \subseteq (X, d)$ is sequentially compact,

then it is closed in (X, d) .

We will eventually show something better:

compact \iff sequentially compact

Lectures 38, 39:

①

→ Def: Let (X, d) be a metric space.

absolutely
dependent
on metric!
Can't be gene-
ralised in
all topological
spaces!

We say that (X, d) is totally bounded if :

$\forall \varepsilon > 0$, we can find finitely many $x_1, \dots, x_n \in X$
 s.t. $X = \bigcup_{i=1}^n B_d(x_i, \varepsilon)$ (We can even consider $n=0$ here,
so that \emptyset is totally bounded
w.r.t. any metric.)

(i.e., if, $\forall \varepsilon > 0$, we can cover X by finitely many
 open balls of radius ε)



As for compactness and sequential compactness :

If (X, d) is a metric space and $K \subseteq X$,

we say that K is totally bounded (in (X, d) ;
usually implied)

if (K, d) is a totally bounded metric space ,

(2)

i.e. if: $\forall \epsilon > 0$, we can find finitely many $x_1, \dots, x_n \in K$,

$$\text{st. } K \subseteq \bigcup_{i=1}^n B_d(x_i, \epsilon) \quad \begin{matrix} \text{open in } \\ \text{balls} \end{matrix}$$

because

the open balls in (K, d) are all

the sets of the form $B_d(x, r) \cap K$, where $x \in K$, $r > 0$.

$\underbrace{\text{open ball}}_{\text{in } (X, d)}$

(Weekly Assignment 13)

→ Observation: Every compact (X, d) is totally bounded.

Proof:

Let $\epsilon > 0$. Then, $X = \bigcup_{x \in X} B_d(x, \epsilon) \quad \begin{matrix} \text{open in } \\ \text{in } (X, d) \end{matrix}$

$\xrightarrow{(X, d) \text{ compact}}$ X can be covered by finitely many of the above open balls.

Since $\epsilon > 0$ was arbitrary, (X, d) is totally bounded. ■



(X, d) totally bounded $\implies (X, d)$ bounded.

→ So, compact \implies bounded
(Second proof).

But the converse is not always true. This sounds very counterintuitive...

(3)

The big theorem we are aiming for is:

Thm: Let (X, d) be a metric space. Then,

the following are **equivalent**:

- i) (X, d) compact.
- ii) (X, d) sequentially compact.
- iii) (X, d) complete and totally bounded.

The order we will prove this in is the following:

compact \Rightarrow sequentially compact \Rightarrow complete and totally bounded



compact.

→ "quite" simple with appropriate setup.

(4)

Compact \Rightarrow sequentially compact

→ T-thm:

Any compact metric space (X, d)
is sequentially compact.



This implies that ; if (X, d) is a metric space,
and K is a compact subset of X ,
then K is sequentially compact.

Proof: Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X .

We will show that there exists a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$, with $x_{k_n} \xrightarrow{d} x$,
for some $x \in X$. Indeed :

Let $Y = \{x_n : n \in \mathbb{N}\}$.

If Y is finite, then $(x_n)_{n \in \mathbb{N}}$ is
eventually constant, thus it converges
in X .

So, we may assume that Y is infinite.

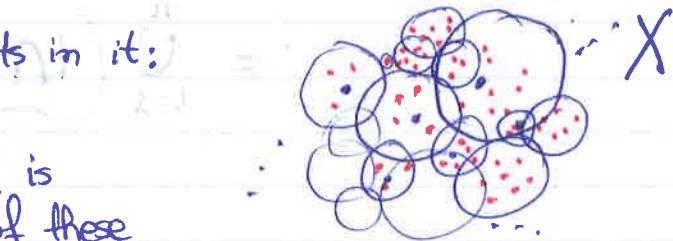
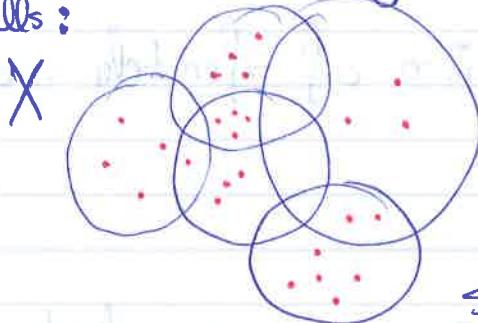
We will show that $\exists x \in X$, s.t. $B_d(x, r) \cap Y$ infinite,
 $r > 0$

then, this x will be arbitrarily close to elements of
our sequence, thus will be the limit of
some subsequence.

for contradiction, suppose that the above is false.

Idea: Then, $\bigcup_{x \in X}$, if some open ball B_x centered at x , with finitely many red dots in it:

But, X compact, so X is the union of finitely many of these balls:



(Say, 100 balls unite to give X)

So, the red dots are exactly the ones in these, finitely many, balls.

Since each of these finitely many balls contains finitely many red dots, there are finitely many red dots in total, contradiction!

Let's make this formal:

Then, $\bigcup_{x \in X}$, there exists an open ball $B_d(x, r_x)$,

with $Y \cap B_d(x, r_x)$ finite.

Since $X = \bigcup_{x \in X} B_d(x, r_x)$, and (X, d) is compact,
 $\underbrace{\qquad}_{\text{open in } (X, d)}$

we have that X is covered by finitely many of these balls; i.e., $x_1, \dots, x_n \in X$ st.
 $X = \bigcup_{i=1}^n B_d(x_i, r_{x_i})$.

(6)

Thus, since $Y \subseteq X$, we have:

$$Y = Y \cap X = Y \cap \left(\bigcup_{i=1}^n B_d(x_i, r_{x_i}) \right) = \\ = \bigcup_{i=1}^n \underbrace{\left(Y \cap B_d(x_i, r_{x_i}) \right)}_{\text{finite}},$$

finite, as a union of finitely many finite sets

Thus, our initial assumption is wrong: which means

that there exists $x \in X$, s.t. $B_d(x, r) \cap Y$ is infinite, $\forall r > 0$.

So:

- for $r=1$, $\exists k_1 \in \mathbb{N}$ s.t. $x_{k_1} \in Y \cap B_d(x, 1)$
- for $r=\frac{1}{2}$, $Y \cap B_d(x, \frac{1}{2})$ is infinite, so $\exists k_2 > k_1$ in \mathbb{N} ,

s.t. $x_{k_2} \in Y \cap B_d(x, \frac{1}{2})$

(as $[Y \cap B_d(x, \frac{1}{2})] \setminus \{x_n : n \leq k_1\}$ is also infinite, thus non-empty)

the first
 k_1 terms of
 $(x_n)_{n \in \mathbb{N}}$



(7)

- for $r = \frac{1}{3}$, $\exists k_3 \in \mathbb{N}$ s.t. $x_{k_3} \in B_d(x, \frac{1}{3}) \cap Y$, $k_3 > k_2$

(as $B_d(x, \frac{1}{3}) \cap Y$ is infinite,
thus we can pick $x_{k_3} \in Y \setminus \{x_{k_1}, x_{k_2}\}$)

Inductively, we construct a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$, with $x_{k_n} \in B_d(x, \frac{1}{n}) \quad \forall n \in \mathbb{N}$,
thus with $x_{k_n} \xrightarrow{d} x$.



If you are interested:

We have just shown the $\boxed{(X, d)}$ is compact \iff every sequence in (X, d) has a subsequence that converges in (X, d) !"

In topological spaces, the (\iff) doesn't hold:
 there is a notion of sequence convergence, and
 the (\Rightarrow) above holds for (X, T) compact.

But (\Leftarrow) doesn't hold: knowing that all sequences in (X, T) have subsequences that converge in (X, T)
is not enough to ensure that every open (w.r.t. T) cover of X has a finite subcover! We need to

(8)

know something stronger: that every net in (X, τ) has a convergent subnet. Nets are generalised notions of sequences, that are complex enough to capture the complexity of a topological space (while sequences are too simple, too poor to achieve that).

- Note that, as in metric spaces we have characterisation of continuity and closed sets via sequences, in topological spaces we have characterisation of continuity and closed sets via nets.

Sequentially compact \Rightarrow complete + totally bounded

→ **Thm:** Any sequentially compact metric space (X, d) is complete and totally bounded.

Proof: $\xrightarrow{\text{trivial}}$

• (Sequentially compact \Rightarrow complete):

We need to show that any Cauchy sequence (w.r.t. d) in X converges (w.r.t. d) in X .

Indeed, let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence (w.r.t. d) in X .
 (Idea: Suffices to find one convergent subsequence!)

Since $(x_n)_{n \in \mathbb{N}}$ lives in X , and (X, d) is sequentially compact,

there exists a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$,

with $x_{k_n} \xrightarrow{d} x$, for some $x \in X$.

Since $(x_n)_{n \in \mathbb{N}}$ is Cauchy, and has a convergent subsequence, $(x_n)_{n \in \mathbb{N}}$ must itself converge to the limit of the subsequence; i.e., $x_n \xrightarrow{d} x$ ($\in X$).

Thus, $(x_n)_{n \in \mathbb{N}}$ converges in X .

→ quite simple by contradiction.

• (Sequentially compact \Rightarrow totally bounded):

Remember: When we know something for sequences, and we want to use this to deduce something, very often contradiction works... Remember for instance, characterisation of continuity via sequences.

(10)

Suppose that (X, d) is not totally bounded.

We will construct a sequence $(x_n)_{n \in \mathbb{N}}$ in X

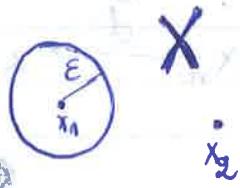
that doesn't have any convergent subsequence
(convergent in (X, d)):

Since (X, d) is not totally bounded; there exists some $\epsilon > 0$
s.t. X can't be covered by finitely many open
balls of radius ϵ ;

i.e., s.t. $X \setminus (\text{union of any finitely many open balls}) \neq \emptyset$.

In particular:

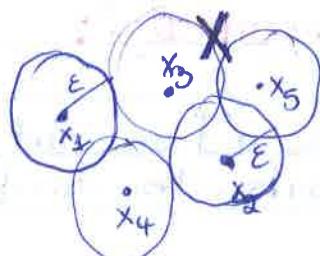
- Pick an $x_1 \in X$ (doesn't matter which)



$X \setminus B_d(x_1, \epsilon) \neq \emptyset$, thus

$\exists x_2 \in X \setminus B_d(x_1, \epsilon)$;

then, $d(x_1, x_2) \geq \epsilon$.



$X \setminus (B_d(x_1, \epsilon) \cup B_d(x_2, \epsilon)) \neq \emptyset$, thus

$\exists x_3 \in X \setminus (B_d(x_1, \epsilon) \cup B_d(x_2, \epsilon))$; then,
 $d(x_3, x_1) \geq \epsilon$, $d(x_3, x_2) \geq \epsilon$. And so on...

We create thus a sequence $(x_n)_{n \in \mathbb{N}}$ in X , with

$$d(x_n, x_m) \geq \epsilon, \quad \forall n \neq m \text{ in } \mathbb{N}.$$

(where our ϵ is a fixed constant > 0).

So, no subsequence of $(x_n)_{n \in \mathbb{N}}$ can converge
(as it cannot be Cauchy).

This is a contradiction, as (X, d) is sequentially compact.

So, our initial assumption was wrong; thus,
 (X, d) is totally bounded.



By now, we have shown that

$$\begin{aligned} (X, d) \text{ compact} &\implies (X, d) \text{ sequentially compact} \implies \\ &\implies (X, d) \text{ complete and } \underbrace{\text{totally bounded}}_{\text{bounded}}. \end{aligned}$$

So, if $K \subseteq (X, d)$, then

$$\begin{aligned} K \text{ compact subset of } (X, d) &\implies (K, d) \text{ complete and bounded (w.r.t. } d) \\ \text{i.e., } (K, d) \text{ compact} & \quad \downarrow \\ \text{"Complete or special case of closed"} & \quad \downarrow \\ K \text{ closed in } (X, d) & \end{aligned}$$

(12)

Thus, this is a ind proof that

K compact in (X, d) $\Rightarrow K$ closed and bounded
 (in (X, d)) (w.r.t. d)

→ hard.

Complete + totally bounded \Rightarrow compact

→ Thm: Any complete and totally bounded metric space (X, d)
 is compact.

Proof: for this proof, we will use the following
 generalisation of the nested intervals theorem;

→ Thm (Cantor): Let (X, d) be a metric space. Then,

(X, d) is complete

← notice that also the
 converse holds!

for any closed, non-empty subsets of X

$F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq \dots$, with $\lim_{n \rightarrow \infty} \text{diam}(F_n) = 0$,

it holds that $\bigcap_{n=1}^{+\infty} F_n =$ a singleton.

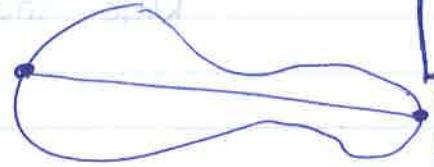
(13)

Note that, for any $F \subseteq (X, d)$,

$$\text{diam}(F) = \sup \{d(x, y) : x, y \in F\}$$

the diameter
of F

morally, the "largest" distance between any two elements of F .



Let us accept Cantor's theorem as a given here.

We want to show that (X, d) is compact; i.e., that any open cover \mathcal{A} of (X, d) has a finite subcover. Indeed:

Let \mathcal{A} be an open cover of (X, d) .

i.e., \mathcal{A} consists
of open sets in X ,
(w.r.t. d) whose union is X .

We want to show that \exists finitely many elements of \mathcal{A} that also cover X .

We consider all the subsets of X that can actually be covered by finitely many elements of \mathcal{A} :

(14)

$$\mathcal{C} := \left\{ K \subseteq X : \exists n \in \mathbb{N} \text{ and } A_1, \dots, A_n \in \mathcal{A} \text{ s.t. } K \subseteq \bigcup_{i=1}^n A_i \right\}.$$

All we want to show is that $X \in \mathcal{C}$.

Note that, if $K_1, \dots, K_m \in \mathcal{C}$, where $m \in \mathbb{N}$, then $K_1 \cup \dots \cup K_m \in \mathcal{C}$ as well

(as, if each K_i can be covered by finitely many elements of \mathcal{A} , then so can the finite union $K_1 \cup \dots \cup K_m$).

I

Claim:

for any F closed subset of (X, d) , such that $F \notin \mathcal{C}$ (i.e., s.t. F can't be covered by finitely many elements of \mathcal{C}), we can find arbitrarily small subsets of F that are closed in (X, d) and not in \mathcal{C} either.

More formally: If $F \subseteq (X, d)$ closed, with $F \notin \mathcal{C}$, for any $\varepsilon > 0$, if $F_\varepsilon \subseteq F$, closed in (X, d) , s.t. $\text{diam}(F_\varepsilon) \leq \varepsilon$ and $F_\varepsilon \notin \mathcal{C}$.

Proof of Claim: Let F closed in (X, d) , $F \notin \mathcal{C}$.

Let $\varepsilon > 0$.

(15)

(X, d) totally bounded $\Rightarrow X = \bigcup_{i=1}^n B_d(x_i, \frac{\epsilon}{2}),$ for some

$x_1, \dots, x_n \in X,$
 $n \in \mathbb{N}.$

$$\Rightarrow f = \left(\underbrace{\bigcup_{i=1}^n B_d(x_i, \frac{\epsilon}{2})}_{\text{X}} \right) \cap F = \bigcup_{i=1}^n \left(\underbrace{B_d(x_i, \frac{\epsilon}{2}) \cap F}_{\substack{\text{closure} \\ \text{in } (X, d)}} \right).$$

||
 $f_i \subseteq F$

Since F is closed, $F = \overline{\bigcup_{i=1}^n F_i}$ (as F contains the limit points of all sequences in the f_i 's that converge in (X, d))

And: $\text{diam}(f_i) \leq \epsilon$ (as f_i is contained in a ball of radius $\frac{\epsilon}{2}$)

$\Rightarrow \text{diam}(\bar{f}_i) \leq \epsilon$ (as \bar{f}_i has distance 0 from f_i),

$\forall i=1, \dots, n.$

Moreover, $\exists i \in \{1, \dots, n\}$ s.t. $\bar{f}_i \notin \mathcal{C}$ (indeed, otherwise $\bar{f}_1, \dots, \bar{f}_n \in \mathcal{C} \Rightarrow F = \bar{f}_1 \cup \dots \cup \bar{f}_n \in \mathcal{C}$, contradiction.)

So, \bar{f}_i satisfies our requirements.

— [End of proof
of Claim]

II Now, we assume that $X \notin \mathcal{C}$. Using this assumption:

By the Claim, we construct a nested sequence

$f_1 \supseteq f_2 \supseteq \dots \supseteq f_m \supseteq \dots$ of closed subsets of (X, d) ,

that are not in \mathcal{C} ,

and $\text{diam}(f_n) \xrightarrow{n \rightarrow \infty} 0$:

- X closed in (X, d) , $X \notin \mathcal{C}$ over assumption

Claim If $f_i \subseteq X$, f_i closed in (X, d) , $f_i \notin \mathcal{C}$,

with $\text{diam}(f_i) < 1$.

- f_1 closed in (X, d) , $f_1 \notin \mathcal{C}$

Claim If $f_2 \subseteq f_1$, f_2 closed in (X, d) , $f_2 \notin \mathcal{C}$,

with $\text{diam}(f_2) < \frac{1}{2}$.

- f_2 closed in (X, d) , $f_2 \notin \mathcal{C}$

Claim If $f_3 \subseteq f_2$, f_3 closed in (X, d) , $f_3 \notin \mathcal{C}$,

with $\text{diam}(f_3) < \frac{1}{3}$.

... And so on.

17

We thus create this way a nested sequence

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots \supseteq F_n \supseteq \dots$$

of closed subsets of (X, d) ,

that are not in \mathcal{C} , with $\text{diam}(F_n) < \frac{1}{n} \ \forall n \in \mathbb{N}$,

thus with $\text{diam}(F_n) \xrightarrow{n \rightarrow \infty} 0$
(by sandwich lemma).

III

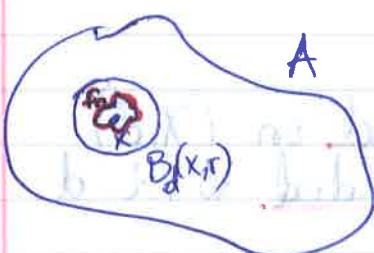
Since (X, d) complete, by Cantor's theorem we deduce that

$$\bigcap_{n=1}^{\infty} F_n = \{x\}, \text{ for some } x \in X.$$

Here is where we use that (X, d) is complete

Since \mathcal{A} covers X , there exists $A \in \mathcal{A}$ s.t. $x \in A$.

Remember: A open!



So,

If open ball $B_d(x, r) \subseteq A$.

Now, $\text{diam}(F_n) \xrightarrow{n \rightarrow \infty} 0$, thus $\exists n \in \mathbb{N}$

s.t. $\text{diam}(F_n) < r$;

and, remember, $x \in F_n$. This means

that $F_n \subseteq B_d(x, r)$ (as, $\forall y \in F_n, d(y, x) \leq \text{diam}(F_n) < r$),

so $F_n \subseteq A$, thus F_n is covered by a

single set in \mathcal{A} , thus $F_n \in \mathcal{C}$, a contradiction!

Thus, our assumption (in ②) that $x \notin \mathcal{C}$ was false. So, $x \in \mathcal{C}$, thus the ^{open} cover \mathcal{U} of X has a finite subcover.

Since the open cover \mathcal{U} of X was arbitrary, (X, d) is compact.

⚠ We have thus shown that

(X, d) compact $\iff (X, d)$ complete and totally bounded.

In particular, if $K \subseteq (X, d)$, then

K compact in (X, d) \implies K closed in (X, d) and bounded w.r.t. d .

But the converse doesn't always hold, as, in general, closed is much weaker than complete, and bounded much weaker than totally bounded.

However, in \mathbb{R}^n , the converse holds! ☺
It is based on the following:

(19)

→ Corollary: Let (X, d) be a complete metric space.

Let $K \subseteq X$.

Then, K compact (in (X, d)) \iff K closed and totally bounded (in (X, d)).

Proof: K compact in (X, d) \iff

\iff (K, d) complete and totally bounded

\iff K closed and totally bounded

in (X, d) .

(X, d) complete,
so (K, d) complete

\iff K closed in (X, d)

Corollary :

$$K \subseteq \mathbb{R}^n$$

equipped
with the
usual metric

is compact

$\Leftrightarrow K$ is closed and bounded

(in $(\mathbb{R}^n, \text{usual metric})$)

Proof: (\mathbb{R}^n, d) is complete.

↓
usual
metric

↑ previous Corollary

So, $K \subseteq \mathbb{R}^n$ compact $\Leftrightarrow K$ closed and totally bounded.

Thus, it suffices to show that, w.r.t. the usual metric,

K is totally bounded $\Leftrightarrow K$ bounded:

(\Rightarrow) always true.

(\Leftarrow) K bounded \Rightarrow there exists some

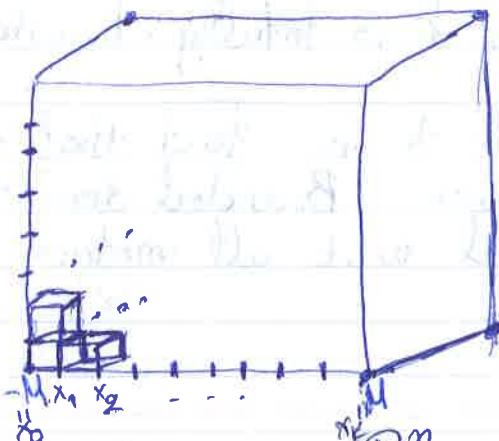
ball $B_d(O, M)$, containing K .

For any $\epsilon > 0$, $B_d(O, M)$ can be

union of finitely many balls of radius ϵ .

Indeed: $B_d(O, M) \subseteq [-M, M]^n$ (a cube).

(21)



Now, any cube in \mathbb{R}^n , of side r , has diameter (length of diagonal) $\sqrt{n} \cdot r$.

Now, we consider a partition of $[-N, N]$, that splits $[-M, M]$ in k intervals

$I_i = [x_{i-1}, x_i]$, $i=1, \dots, k$, of equal length, $< \frac{\epsilon}{\sqrt{n}}$.

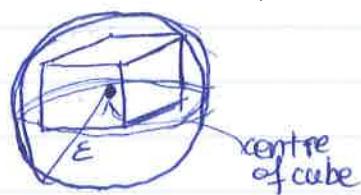
Now, $[-N, N]^n =$ the union of all the cubes of the form $I_{i_1} \times I_{i_2} \times \dots \times I_{i_n}$,

where $i_1, i_2, \dots, i_n \in \{1, \dots, k\}$.

these are finitely many cubes, of diameter $< \frac{\epsilon}{\sqrt{n}} = \epsilon$;

thus, each is contained in an open ball in \mathbb{R}^n , of radius ϵ .

So, $[-N, N]^n$ is covered by finitely many balls in \mathbb{R}^n , of radius $\epsilon > 0$.



Since $\epsilon > 0$ was arbitrary, K is totally bounded

(remember, this is due to the fact that our metric is the usual one. Bounded doesn't imply totally bounded w.r.t. all metrics...).



Thm (Heine-Borel): In \mathbb{R}^n , with the usual metric,
 all rectangles $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$,
 where $a_i, b_i \in \mathbb{R}$ $i=1, \dots, n$, and $a_i \leq b_i$ $i=1, \dots, n$,
 are compact.

Proof: Rectangles are closed and bounded in \mathbb{R}^n .



Note that, for $n=1$, the above means that

closed intervals $[a, b]$ in \mathbb{R} are compact
 (w.r.t. the usual metric).